Distributed $\mathcal{L}_2$ -gain Output-Feedback Control of Homogeneous and Heterogeneous Systems [★]

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Abstract

The importance of static output feedback (OPFB) design for aircraft control, process control, and elsewhere has been well documented since the 1960s, since full state measurements are not usually available in practical systems. This problem is compounded in the case of multi-agent systems (MAS) where each agent has its own state variable and measured outputs. Therefore, this paper addresses the $\mathcal{L}_2$ -gain OPFB synchronization of linear MAS subject to external disturbances. Both homogeneous and heterogeneous MAS are considered. For homogeneous MAS, it is shown that the $\mathcal{L}_2$ -gain static OPFB synchronization problem of MAS can be cast into the $\mathcal{L}_2$ -gain static OPFB problem for a set of decoupled systems that depend on the graph topology. A modified Riccati equation is introduced which gives the OPFB gain and the coupling gain of the proposed static OPFB control protocol. For heterogeneous MAS, it is shown that the $\mathcal{L}_2$ -gain synchronization problem can be cast into the $\mathcal{L}_2$ -gain static OPFB problem of a set of decoupled systems plus a coupling condition on their dynamic compensators that depends on the graph topology. A certain novel gain matrix is introduced in the dynamics of the control protocol to improve the performance.

Key words: Multi-agent systems, $\mathcal{L}_2$ -gain synchronization, LQR based optimal design, output-feedback control.

1 Introduction

Distributed control of multi-agent systems (MAS) on communication graphs has been well studied in the literature [1–6], due to its potential applications in a variety of engineering systems. A control law, which depends only on the local neighbor information, is designed for each agent to make the network of agents converge to a common value of interest. If the common value that agents agree on is not prescribed, the problem is called regulation, and if all agents follow the trajectory of a leader node, the problem is known as leader-follower control [7, 8].

The design of distributed control protocols for MAS with general linear dynamics has been considered in the literature [9–11]. However, attenuating the effect of the disturbance on the system performance is ignored in most existing methods. The disturbance rejection for MAS has been formulated as an H_∞ control problem in [12–14] under a leaderless framework. Conditions in terms of linear matrix inequalities were given to ensure consensus with a desired H_∞ performance. Existing methods, however, mostly require the knowledge of the relative states between neighboring agents, which may not be available for measurement.

The importance of output feedback (OPFB) design, particularly static output feedback [15], for aircraft control [16], optimal process control [17], and elsewhere has been well documented since the 1960s, since full state measurements are not usually available in practical systems. This problem is compounded in the case of MAS where each agent has its own state variable and measured outputs. Therefore, this paper addresses the $\mathcal{L}_2$ -gain OPFB synchronization of linear MAS subject to external disturbances.

To obviate the requirement for full state measurements, distributed observer-based OPFB and distributed dynamic OPFB protocols were derived in [18–20] for H_∞ control. For the leader-follower systems, distributed state-feedback protocols were designed in [21, 22], and a distributed observer-type protocol was presented in [23]. The above techniques assumed homogeneous MAS, that is, all agents have identical dynamics. The heterogeneous case, where agents can have different dynamics, has been considered by [24–26]. It is shown that each agent and its controller must contain an internal copy of the leader's dynamics. This work is based either on the output regulator equations [27, 28] or the idea of p-copy [27]. Some works have been done on H_∞ control for heterogeneous systems [29–33].

Although several OPFB control protocols are designed for homogeneous MAS in the presence of disturbance, existing results are mainly dynamic OPFB methods. The static OPFB problem for single-agent systems is one of the most researched problems in the control society. The use of static OPFB allows flexibility and simplicity of implementation, and it is of extreme importance in practical controller design applications [16, 17]. For disturbance-

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free case, decentralized static OPFB stabilization and synchronization of networks were studied in [34]. Static OPFB protocols are derived for consensus [35] and synchronization of homogeneous MAS in the presence of disturbances [36]. Moreover, for heterogeneous MAS, the design of a dynamic OPFB protocol has been considered for the case where the disturbances in the dynamics of the follower agents are generated by the leader.

In this paper, the distributed control of leader-follower problem subject to external disturbances using OPFB is considered. A complete approach to OPFB design is given for both homogeneous and heterogeneous MAS. For the homogeneous MAS, it is shown that this problem can be cast into the $\mathcal{L}_2$ -gain static OPFB problem for a set of decoupled systems. A modified algebraic Riccati equation (ARE), which takes into account the spectrum property of the communication topology, is introduced. For the heterogeneous MAS, a modified dynamic OPFB control protocol is employed to solve the bounded $\mathcal{L}_2$ -gain synchronization problem. It is shown that the $\mathcal{L}_2$ -gain synchronization problem of heterogeneous MAS can be cast into the $\mathcal{L}_2$ -gain static OPFB problem of a set of decoupled systems plus a coupling condition on their dynamic compensators that depends on the graph topology. It is also shown that the gains of the dynamic compensator can be found by solving AREs for each agent.

2 Preliminaries and Problem Formulation

Throughout this paper, the following notations are used: A matrix $T \in \mathbb{C}^{n \times n}$ is unitary if $T^*T = TT^* = I_n$. $\mathbf{1}_n \in \mathbb{R}^n$ denotes the vector whose elements are equal to 1. $\mathcal{L}_2[0, \infty)$ denotes the space of square integrable vector functions over $[0, \infty)$. $A \otimes B$ denotes the Kronecker product of matrices A and B . $\text{diag}(A_1, \dots, A_n)$ represents a block diagonal matrix with matrices $A_i, i = 1, \dots, n$, on its diagonal.

2.1 Graph Theory

A directed graph $\mathcal{G}$ consists of a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\alpha_1, \dots, \alpha_N\}$ is a finite nonempty set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of ordered pairs of nodes, called edges. Denote the adjacency matrix as $\mathcal{A} = [a_{ij}]$ with $a_{ij} > 0$ if $(\alpha_i, \alpha_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The set of nodes α_j with edges incoming to node α_i is called the neighbors of node i , namely $\mathcal{N}_i = \{\alpha_j : (\alpha_j, \alpha_i) \in \mathcal{E}\}$. The graph Laplacian matrix is defined as $L = D - \mathcal{A}$, which has all row sums equal to zero. $D = \text{diag}(d_i)$ is called the in-degree matrix, where $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$ is the weighted in-degree of node i . A (directed) tree is a connected digraph where every node except one, called the root, has in-degree equal to one. A graph has a spanning tree if a subset of the edges forms a directed tree.

Assumption 1 The digraph $\mathcal{G}$ contains a spanning tree and the leader is pinned to a root node.

2.2 $\mathcal{L}_2$ -gain OPFB Synchronization for Homogeneous Multi-agent Systems

Consider N identical linear dynamic systems given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + Dw_i \\ y_i &= Cx_i \end{aligned} \quad i = 1, \dots, N, \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^p$, $w_i(t) \in \mathbb{R}^m$ and $y_i(t) \in \mathbb{R}^q$ are the state, control input, external disturbance and output of node i , respectively. A, B, C and D are constant matrices with C full row rank. Let the leader dynamics be

$$\dot{x}_0 = Ax_0, \quad y_0 = Cx_0 \quad (2)$$

where $x_0(t) \in \mathbb{R}^n$ is leader state and $y_0(t) \in \mathbb{R}^q$ is the measured output.

Define the local neighborhood error of the node i as

$$\varepsilon_{y_i} = \sum_{j \in \mathcal{N}_i} a_{ij}(y_j - y_i) + g_i(y_0 - y_i) \quad (3)$$

where $g_i \geq 0$ is called the pinning gain, with $g_i > 0$ for a small subset of agents having direct access to the leader. Consider the static OPFB controller [36] for each node i as

$$u_i = cK\varepsilon_{y_i} \quad (4)$$

where $c > 0$ is the coupling gain, and $K \in \mathbb{R}^{p \times q}$ is the feedback control gain matrix to be determined.

The synchronization error for agent i is defined as

$$\delta_i = x_i - x_0 \quad (5)$$

Let $x = [x_1^T, \dots, x_N^T]^T$, $\underline{x}_0 = (\mathbf{1}_N \otimes I_n)x_0 \in \mathbb{R}^{nN}$, $\delta = [\delta_1^T, \dots, \delta_N^T]^T$, $\omega = [\omega_1^T, \dots, \omega_N^T]^T$ and $\varepsilon_y = [\varepsilon_{y1}^T, \dots, \varepsilon_{yN}^T]^T$. Then the global synchronization dynamics become

$$\begin{aligned} \dot{\delta} &= (I_N \otimes A - c(L + G) \otimes BKC)\delta + (I_N \otimes D)\omega \\ \varepsilon_y &= -((L + G) \otimes C)\delta \end{aligned} \quad (6)$$

where $G = \text{diag}(g_1, \dots, g_N)$ is the diagonal matrix of pinning gains. Since L has all row sums equal to zero, one has $L\mathbf{1} = 0$ with $\mathbf{1} = [1, \dots, 1]^T$. Moreover, under Assumption 1, $(L + G)$ is positive definite.

Problem 1 (Bounded $\mathcal{L}_2$ -gain static OPFB synchronization problem for homogeneous MAS). Given N identical systems in (1) and a prescribed $\gamma > 0$, design a distributed static OPFB protocol (4) such that

- (1) The global synchronization error dynamics (6) is asymptotically stable.
- (2) The following bounded $\mathcal{L}_2$ -gain condition is satisfied for any non-zero $\omega \in \mathcal{L}_2[0, \infty)$, under zero initial conditions $x_i(0) = x_0(0) = 0$.

$$\int_0^\infty \varepsilon_y^T \varepsilon_y dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt \quad (7)$$

where γ is the disturbance attenuation level.

2.3 $\mathcal{L}_2$ -gain OPFB Synchronization for Heterogeneous Multi-agent Systems

Consider N heterogeneous linear dynamic systems with state realization as

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + D_i w_i \\ y_i &= C_i x_i \end{aligned} \quad i = 1, \dots, N. \quad (8)$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{p_i}$, $w_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^q$ are the state, control input, external disturbance and output of node i , respectively, with n_i, p_i, m_i and q as their dimensions. The dynamics are heterogeneous in the sense that A_i, B_i, C_i and D_i may be different for all agents.

Let the leader dynamics be given by

$$\dot{\zeta}_0 = S\zeta_0, \quad y_0 = \bar{R}\zeta_0 \quad (9)$$

where $\zeta_0(t) \in \mathbb{R}^m$, $y_0(t) \in \mathbb{R}^q$, $S \in \mathbb{R}^{m \times m}$ and $\bar{R} \in \mathbb{R}^{q \times m}$. Define

$$e_i = y_i - y_0 \quad (10)$$

as the local output tracking error for agent i and $e = [e_1^T, \dots, e_N^T]^T$ as the global output tracking error.

Consider the dynamic OPFB control protocol as

$$u_i = K_i(y_i - C_i \Pi_i \zeta_i) + \Gamma_i \zeta_i$$

$$\dot{\zeta}_i = S \zeta_i + cF \left(\sum_{j \in \mathcal{N}_i} a_{ij}(\zeta_j - \zeta_i) + g_i(\zeta_0 - \zeta_i) \right) \quad (11)$$

where $c > 0$ is the coupling gain, $K_i \in \mathbb{R}^{p_i \times n_i}$ and $F \in \mathbb{R}^{m \times m}$ are the OPFB gain matrices to be determined. Moreover, matrices $\Pi_i \in \mathbb{R}^{n_i \times m}$ and $\Gamma_i \in \mathbb{R}^{p_i \times m}$ satisfy the following regulator equations [27, 28]

$$A_i \Pi_i + B_i \Gamma_i = \Pi_i S; \quad C_i \Pi_i = \bar{R} \quad i = 1, \dots, N \quad (12)$$

It is shown in [24, 25] that the existence of the solution to (12) is necessary and sufficient for solvability of the output synchronization problem. Conditions for existence of solution (Π_i, Γ_i) to (12) are well-known in the output regulator problem of single-agent system [27, 28].

Remark 1 The synchronized reference generator in [24] was given by

$$\dot{\zeta}_i = S \zeta_i + \sum_{j \in \mathcal{N}_i} a_{ij}(\zeta_j - \zeta_i)$$

which does not incorporate the gain cF , in contrast to our approach, (see (11)), and thus the spectrum of the leader's dynamics has to be purely imaginary.

Remark 2 In [25], the authors introduced a dynamic full information feedback control protocol as:

$$u_i = K_{1i} x_i + K_{2i} \zeta_i$$

where ζ_i is generated by (11) with $F = I$. Then, the gain c in (11) depends on the spectrum of the leader's dynamics which might be unstable. By contrast, in our method, c will turn out in Theorem 4 to depend only on the spectrum of the communication graph. That is, including F in (11) improves the design performance.

Problem 2 (Bounded $\mathcal{L}_2$ -gain OPFB synchronization problem for heterogeneous MAS). Given N non-identical systems in (8) and a prescribed $\gamma > 0$, design a distributed dynamic OPFB protocol (11) such that

- (1) The N systems in (8) reach synchronization in the sense of $\|y_i(t) - y_0(t)\| \rightarrow 0$, $i = 1, \dots, N$, when $\omega_i = 0$.
- (2) The following bounded $\mathcal{L}_2$ -gain condition is satisfied for any non-zero $\omega \in \mathcal{L}_2[0, \infty)$, under zero initial conditions $x_i(0) = \zeta_i(0) = \zeta_0(0) = 0$.

$$\int_0^\infty e^T e dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt \quad (13)$$

where γ is the disturbance attenuation level.

3 Solution to $\mathcal{L}_2$ -gain OPFB Synchronization of Homogeneous Multi-agent Systems

This section presents a solution to Problem 1. The following theorem is inspired by [11].

Theorem 1 Make Assumption 1 and assume $(L + G)$ is a normal matrix. For a given $\gamma > 0$, there exists a static OPFB protocol (4) which solves the bounded $\mathcal{L}_2$ -gain synchronization Problem 1, if and only if the N systems

$$\begin{aligned} \dot{\hat{\delta}}_i &= (A - c\lambda_i BKC)\hat{\delta}_i + D\hat{\omega}_i \\ \hat{\varepsilon}_{y_i} &= -\lambda_i C\hat{\delta}_i \quad i = 1, \dots, N \end{aligned} \quad (14)$$

are simultaneously asymptotically stable and the $\mathcal{L}_2$ -gain of systems (14) are all bounded by γ , where λ_i , $i = 1, \dots, N$ are the nonzero eigenvalues of $(L + G)$.

Proof. Since $(L + G)$ is normal, there exists a unitary transformation matrix T such that

$$T(L + G)T^* = \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_N\} \quad (15)$$

Let $\hat{\delta} = (T \otimes I_n)\delta$. Then, one has

$$\begin{aligned} \dot{\hat{\delta}} &= (I_N \otimes A - cT(L + G)T^* \otimes BKC)\hat{\delta} + (T \otimes D)\omega \\ \varepsilon_y &= -((L + G)T^* \otimes C)\hat{\delta} \end{aligned} \quad (16)$$

Now define the following transformations

$$\omega = (T^* \otimes I_m)\hat{\omega}, \quad \hat{\varepsilon}_y = (T \otimes I_q)\varepsilon_y \quad (17)$$

Then, (16) becomes

$$\begin{aligned} \dot{\hat{\delta}} &= (I_N \otimes A - c\Lambda \otimes BKC)\hat{\delta} + (I_N \otimes D)\hat{\omega} \\ \hat{\varepsilon}_y &= -(\Lambda \otimes C)\hat{\delta} \end{aligned} \quad (18)$$

which gives (14). It now remains to show that the $\mathcal{L}_2$ -gain of (18) is equal to that of (6). Since T is unitary, one has

$$\|\hat{\varepsilon}_y\|_2^2 = \int_0^\infty \varepsilon_y^T (T \otimes I_q)^* (T \otimes I_q) \varepsilon_y dt = \|\varepsilon_y\|_2^2 \quad (19)$$

Similarly, one can show that $\|\hat{\omega}\|_2^2 = \|\omega\|_2^2$. ■

Remark 3 Theorem 1 requires that $(L + G)$ be a normal matrix. The set of graphs with normal $(L + G)$ includes the undirected graphs discussed in [14], but is more general since the eigenvalues of $(L + G)$ can be complex. In [36], the global design of synchronizing controllers is studied and it is shown that there exists a global optimal controller if a condition similar to normality is satisfied. Therefore, this class of graphs evidently arises naturally in the studying of global optimal synchronization performance.

The next result shows how to select the static OPFB control gain K and the coupling gain c in (4) by using the LQR based optimal design to solve Problem 1. It extends the results of [11] for the OPFB case and [37] to the multi-agent case.

Theorem 2 Make Assumption 1, assume $(L + G)$ is a normal matrix, (A, B) is stabilizable and (A, C) is detectable. Let the eigenvalues of $(L + G)$ be $\lambda_i = \text{Re}(\lambda_i) + j\text{Im}(\lambda_i)$, $i = 1, \dots, N$ and $R = R^T > 0$. The systems (14) are asymptotically stable with $\mathcal{L}_2$ -gain bounded by $\gamma > 0$, or equivalently Problem 1 is solved, if there exist matrices K, M and $P = P^T > 0$ such that

$$KC = R^{-1}(B^T P + M) \quad (20)$$

$$\begin{aligned} PA + A^T P + \lambda_M^2 C^T C + \frac{1}{\gamma^2} PDD^T P - PBR^{-1}B^T P \\ + c\lambda_M M^T R^{-1}M = 0 \end{aligned} \quad (21)$$

where $\underline{\lambda}_M = \max_{i=1,\dots,N} \{Re(\lambda_i)\}$, $\lambda_M = \max_{i=1,\dots,N} \{|\lambda_i|\}$ and the coupling gain c satisfies

$$c \geq \frac{1}{\underline{\lambda}_m} \quad (22)$$

with $\underline{\lambda}_m = \min_{i=1,\dots,N} \{Re(\lambda_i)\}$.

Proof. Consider the Lyapunov function candidate

$$V = \hat{\delta}_i^T P \hat{\delta}_i \quad (23)$$

Taking the derivative of V along the trajectory of (14) (setting $\omega_i = 0$), one has

$$\dot{V} = \hat{\delta}_i^T \underbrace{[(A - c\lambda_i BKC)^* P + P(A - c\lambda_i BKC)]}_{\Delta} \hat{\delta}_i \quad (24)$$

where

$$\begin{aligned} \Delta = & (A - BKC)^T P + P(A - BKC) - (cRe(\lambda_i) - 1) \\ & C^T K^T RKC - (cRe(\lambda_i) - 1) PBR^{-1} B^T P + (cRe(\lambda_i) \\ & - 1) M^T R^{-1} M - jcIm(\lambda_i) (C^T K^T M - M^T KC) \end{aligned} \quad (25)$$

Using (20) and (21), one has

$$\begin{aligned} (A - BKC)^T P + P(A - BKC) = & -\lambda_M^2 C^T C - C^T K^T \\ & RKC - \frac{1}{\gamma^2} PDD^T P - (c\underline{\lambda}_M - 1) M^T R^{-1} M \end{aligned} \quad (26)$$

Then, (24) can be further written as

$$\begin{aligned} \dot{V} = & \hat{\delta}_i^T [-\lambda_M^2 C^T C - \frac{1}{\gamma^2} PDD^T P - cRe(\lambda_i) C^T K^T RKC \\ & - (cRe(\lambda_i) - 1) PBR^{-1} B^T P + c(Re(\lambda_i) - \underline{\lambda}_M) \\ & M^T R^{-1} M - jcIm(\lambda_i) (C^T K^T M - M^T KC)] \hat{\delta}_i \end{aligned} \quad (27)$$

Since $P > 0$, $R > 0$ and $\gamma > 0$, $A - c\lambda_i BKC$ is Hurwitz if $c \geq 1/\underline{\lambda}_m$. Note that the last term of (27) is zero since $C^T K^T M - M^T KC$ is a skew symmetric matrix.

We now show that $\mathcal{L}_2$ -gain of systems (14) are all bounded by γ . Consider the Lyapunov function $V = \hat{\delta}_i^T P \hat{\delta}_i$ and define a Hamiltonian function as

$$H = \frac{dV}{dt} + \hat{\varepsilon}_{y_i}^T \hat{\varepsilon}_{y_i} - \gamma^2 \hat{\omega}_i^T \hat{\omega}_i \quad (28)$$

Differentiating the Lyapunov function along with (14), (28) becomes

$$\begin{aligned} H = & \hat{\delta}_i^T [(A - c\lambda_i BKC)^* P + P(A - c\lambda_i BKC)] \hat{\delta}_i \\ & + 2\hat{\delta}_i^T P D \hat{\omega}_i + |\lambda_i|^2 \hat{\delta}_i^T C^T C \hat{\delta}_i - \gamma^2 \hat{\omega}_i^T \hat{\omega}_i \end{aligned} \quad (29)$$

Substituting (27) into (29) gives

$$\begin{aligned} H = & \hat{\delta}_i^T [-cRe(\lambda_i) C^T K^T RKC - (cRe(\lambda_i) - 1) \\ & PBR^{-1} B^T P + c(Re(\lambda_i) - \underline{\lambda}_M) M^T R^{-1} M \\ & - jcIm(\lambda_i) (C^T K^T M - M^T KC) + (|\lambda_i|^2 - \lambda_M^2) C^T C] \hat{\delta}_i \\ & - \frac{1}{\gamma^2} \hat{\delta}_i^T PDD^T P \hat{\delta}_i + 2\hat{\delta}_i^T P D \hat{\omega}_i - \gamma^2 \hat{\omega}_i^T \hat{\omega}_i \end{aligned} \quad (30)$$

The last three terms of (30) become $-\gamma^2 \|\hat{\omega}_i - \tilde{\omega}_i\|^2$, where $\tilde{\omega}_i = \frac{1}{\gamma^2} D^T P \hat{\delta}_i$. Therefore, if the coupling gain c satisfies (22), one has $H \leq 0$. Integrating the Hamiltonian function (under zero initial condition $\hat{\delta}_i(0) = 0$) and noting that nonnegativity of the Lyapunov function implies $V(\hat{\delta}_i(T)) > 0, \forall T$, one obtains

$$\int_0^T \hat{\varepsilon}_{y_i}^T \hat{\varepsilon}_{y_i} dt \leq \int_0^T \gamma^2 \hat{\omega}_i^T \hat{\omega}_i dt \quad (31)$$

This completes the proof. $\blacksquare$

Remark 4 The ARE (21) has a positive definite solution P only if $\gamma > \gamma^*$, where γ^* is the smallest allowable γ that guarantees the ARE (21) to have a positive definite solution. This γ^* is larger than the H_∞ gain of the system obtained by state feedback methods, which is computed in [38], as the static OPFB is a subset of the state feedback.

Remark 5 Note that if $M = 0$ in (20) and (21), one has

$$KC = R^{-1} B^T P \quad (32)$$

$$\begin{aligned} PA + A^T P + \lambda_M^2 C^T C + \frac{1}{\gamma^2} PDD^T P \\ - PBR^{-1} B^T P = 0 \end{aligned} \quad (33)$$

In this case, one can say $u = -Ky$ is replaced with $u = -KCx$ and KC is the global optimal state-feedback gain. However, this OPFB problem might not have a global optimal solution, that is, there might be no K to satisfy (32) and (33), whereas the more general equations (20) and (21) may have a solution for some M . Comparing (32) and (33) to (20) and (21), one can conclude that the solution to (20) and (21) gives a suboptimal OPFB gain K . This relationship motivates proposing an algorithm by iterating on a Lyapunov equation until a matrix M is found that satisfies (20) and (21). The matrix M is arbitrary and it shows in some sense the difference between the global optimal state-feedback gain and the proposed OPFB gain.

Similar to the single-agent OPFB case [37], the following algorithm can be used to find the OPFB gain. In fact, the Lyapunov equation (26) is used to evaluate the value function, i.e., to find the P^i , for a given gain K^i . The gains are then updated in the next step to assure condition (20) and (21) are satisfied after convergence.

Algorithm 1: Iterative Solution for OPFB Gain
Start with an admissible gain K^0 and $M = 0$ and set

(1) given K^i , find the P^i using the equation

$$\begin{aligned} (A - BK^i C)^T P^i + P^i (A - BK^i C) + Q \\ + C^T (K^i)^T R (K^i) C - \frac{1}{\gamma^2} P^i D D^T P^i = 0 \end{aligned} \quad (34)$$

with $Q = \lambda_M^2 C^T C + (c\underline{\lambda}_M - 1) (M^i)^T R^{-1} (M^i)$.

(2) update the control policy and the matrix M using

$$\begin{aligned} K^{i+1} = & R^{-1} (B^T P^i + M^i) C^T (C C^T)^{-1} \\ M^{i+1} = & R K^{i+1} C - B^T P^i. \end{aligned} \quad (35)$$

Note that based on (22), $c\underline{\lambda}_M \geq 1$ and thus $Q > 0$.

4 Solution to $\mathcal{L}_2$ -gain OPFB synchronization of heterogeneous Multi-agent systems

In this section, a solution to Problem 2 for heterogeneous MAS is provided. Define the local tracking error as

$$\varepsilon_i = x_i - \Pi_i \zeta_i \quad (36)$$

with $\varepsilon_i(t) \in \mathbb{R}^{n_i}$. The global tracking error is then defined as $\varepsilon = [\varepsilon_1^T, \dots, \varepsilon_N^T]^T \in \mathbb{R}^{\bar{n}}$ with $\bar{n} = n_1 + \dots + n_N$. Define also the local reference synchronization error as

$$\eta_i = \zeta_i - \zeta_0 \quad (37)$$

with $\eta_i(t) \in \mathbb{R}^m$. The global reference synchronization error is then defined as $\eta = [\eta_1^T, \dots, \eta_N^T]^T \in \mathbb{R}^{mN}$. Now, let $x_c = [\varepsilon^T, \eta^T]^T$, $\Omega = \text{diag}(\Omega_i)$ ($\Omega = A, B, C, D, K, \Pi$), $i = 1, \dots, N$. Then the global system error dynamics become

$$\dot{x}_c = A_c x_c + D_c \omega, \quad e = C_c x_c \quad (38)$$

where

$$A_c = \begin{pmatrix} A + BKC & \Pi(c(L + G) \otimes F) \\ 0 & I_N \otimes S - c(L + G) \otimes F \end{pmatrix} \\ D_c = \begin{pmatrix} D \\ 0 \end{pmatrix} \quad C_c = \begin{pmatrix} C & \bar{R}_c \end{pmatrix} \quad (39)$$

and $\omega = [\omega_1^T, \dots, \omega_N^T]^T$, $\bar{R}_c = I_N \otimes \bar{R}$. The following theorem presents a necessary and sufficient condition for Problem 2.

Theorem 3 *Make Assumption 1. For a given $\gamma > 0$, there exists a control protocol in form of (11), (12) which solves the bounded $\mathcal{L}_2$ -gain synchronization Problem 2, if and only if the N systems*

$$\begin{aligned} \dot{\hat{\varepsilon}}_i &= (A_i + B_i K_i C_i) \hat{\varepsilon}_i + D_i \hat{\omega}_i \\ e_i &= C_i \hat{\varepsilon}_i \quad i = 1, \dots, N \end{aligned} \quad (40)$$

are simultaneously asymptotically stable and have $\mathcal{L}_2$ -gain bounded by $\gamma > 0$, and the N systems

$$\dot{\hat{\eta}}_i = (S - c\lambda_l F) \hat{\eta}_i \quad i = 1, \dots, N \quad (41)$$

are simultaneously asymptotically stable, where λ_l , $l = 1, \dots, N$ are the nonzero eigenvalues of $(L + G)$.

Proof. There exists a transformation matrix T such that

$$T(L + G)T^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \quad (42)$$

where Λ is upper triangular. Let

$$\hat{x}_c = \begin{pmatrix} \hat{\varepsilon} \\ \hat{\eta} \end{pmatrix} = \begin{pmatrix} I_{\bar{n}} & 0 \\ 0 & T \otimes I_m \end{pmatrix} x_c \quad (43)$$

Then, from (38) and (39), it can be verified by invoking (42) that $\hat{x}_c$ evolves according to the dynamics

$$\dot{\hat{x}}_c = A'_c \hat{x}_c + D_c \omega, \quad e = C'_c \hat{x}_c \quad (44)$$

where

$$A'_c = \begin{pmatrix} A + BKC & \Pi(c(L + G))T^{-1} \otimes F \\ 0 & I_N \otimes S - cT(L + G)T^{-1} \otimes F \end{pmatrix} \\ C'_c = \begin{pmatrix} C & T^{-1} \otimes \bar{R} \end{pmatrix}$$

Note that the state-space transformation does not change the transfer function. Moreover, the $\mathcal{L}_2$ -gain of (44) is the same as the $\mathcal{L}_2$ -gain of (38). To complete the proof, it remains to show that (44) is stable and satisfies (13), if and only if conditions of Theorem 3 are satisfied. Note that, from (44) one can conclude

$$\dot{\hat{x}}_{c_i} = \begin{pmatrix} \hat{\varepsilon}_i \\ \hat{\eta}_i \end{pmatrix} = A'_{c_i} \hat{x}_{c_i} + D_{c_i} \omega_i, \quad e_i = C_{c_i} \hat{x}_{c_i} \quad (45)$$

where

$$A'_{c_i} = \begin{pmatrix} A_i + B_i K_i C_i & * \\ 0 & S - c\lambda_l F \end{pmatrix} \\ D_{c_i} = \begin{pmatrix} D_i \\ 0 \end{pmatrix} \quad C'_{c_i} = \begin{pmatrix} C_i & * \end{pmatrix}$$

Since the input does not affect $\hat{\eta}_i$, if we choose F to make $S - c\lambda_l F$ stable, then $*$ block does not appear in the transfer function from ω_i to e_i . Thus, if systems (40) are simultaneously asymptotically stable and have $\mathcal{L}_2$ -gain bounded by $\gamma > 0$, and the systems (41) are simultaneously stable, the systems in (45) are asymptotically stable and have $\mathcal{L}_2$ -gain bounded by $\gamma > 0$. Therefore, the sufficiency is proved. The necessity holds obviously. This completes the proof. ■

The following theorem shows how to choose the control parameters K_i , c and F in (11) to satisfy conditions of Theorem 3 and therefore solve the Problem 2. It extends the results of [37, 39] for single-agent system to the multi-agent systems.

Theorem 4 *Make Assumption 1 and assume (A_i, B_i) are stabilizable and (C_i, A_i) are detectable $\forall i = 1, \dots, N$. Let $R_i = R_i^T > 0$, $R = R^T > 0$. The networked systems (40), (41) are asymptotically stable with $\mathcal{L}_2$ -gain bounded by $\gamma > 0$, or equivalently Problem 2 is solved, if $\forall i = 1, \dots, N$, there exist gains K_i , M_i , F , $P_i = P_i^T > 0$, $P = P^T > 0$, $Q = Q^T > 0$ such that*

$$K_i C_i = R_i^{-1} (B_i^T P_i + M_i) \quad (46)$$

$$P_i A_i + A_i^T P_i + C_i^T C_i + \frac{1}{\gamma^2} P_i D_i D_i^T P_i - P_i B_i R_i^{-1} B_i^T P_i + M_i^T R_i^{-1} M_i = 0 \quad (47)$$

$$F = R^{-1} P \quad (48)$$

$$PS + S^T P + Q - PR^{-1}P = 0 \quad (49)$$

with the coupling gain c satisfies

$$c \geq \frac{1}{2\lambda_m} \quad (50)$$

where $\lambda_m = \min_{l=1, \dots, N} \{Re(\lambda_l)\}$.

Proof. Based on Theorem 3, K_i is chosen to make the systems (40) asymptotically stable with $\mathcal{L}_2$ -gain bounded by $\gamma > 0$. This requirement is satisfied if we choose K_i as in (46), (47) (refer to [37, 39]). Moreover, based on Theorem 3, the control gain F and the coupling gain c are chosen to make (41) asymptotically stable for all λ_l , $l = 1, \dots, N$. Considering (48) and (49) one has

$$P(S - c\lambda_l F) + (S - c\lambda_l F)^* P = -Q - (2cRe(\lambda_l) - 1)PR^{-1}P \quad (51)$$

Since $P > 0$ and $Q > 0$, by Lyapunov theory, $(S - c\lambda_1 F)$ is asymptotically stable if the condition (50) is satisfied. This completes the proof. ■

A necessary condition for the existence of a positive definite solution P_i to (47) is that

$$\gamma > \gamma_i^*, \forall i \quad (52)$$

where γ_i^* is the the smallest allowable γ_i that guarantees the ARE (47) to have a positive definite solution P_i .

Remark 6 Note that the same algorithm as Algorithm 1 can be designed for each agent to find the solution to the OPFB gain K_i in (46) and (47). The only difference is that the Riccati equation (47) does not depend on the graph structure and thus $Q_i = C_i^T C_i$ in (34).

5 Simulation Results

Fig. 1. Communication Graph of the Multi-agent System.

Fig. 1 shows the structure of a graph with one leader (node 0) and five follower agents. The followers' dynamics are assumed as [24]

$$\begin{aligned} \dot{x}_i &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c_i \\ 0 & -d_i & -a_i \end{pmatrix} x_i + \begin{pmatrix} 0 \\ 0 \\ b_i \end{pmatrix} u_i + \begin{pmatrix} 0 \\ 0 \\ e_i \end{pmatrix} \omega_i \\ y_i &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x_i \quad i = 1, 2, 3, 4, 5 \end{aligned}$$

with parameters $a_i, b_i, c_i, d_i, e_i > 0$. The specific parameters for each agent is chosen as $\{2, 1, 1, 10, 1\}$, $\{2, 1, 1, 3, 1\}$, $\{2, 2, 1, 10, 1\}$, $\{2, 1, 1, 10, 3\}$, $\{2, 2, 1, 10, 3\}$, respectively. The leader node 0 has the dynamics as

$$\dot{\zeta}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \zeta_0, y_0 = \begin{pmatrix} 1 & 0 \end{pmatrix} \zeta_0$$

Let $Q = I_2$, $R = \text{diag}(2, 3)$. The coupling gain c is chosen as 4.8 and F found by solving the ARE (49) is

$$F = \begin{pmatrix} 0.6210 & 0.4142 \\ 0.2761 & 0.8782 \end{pmatrix}. \text{ The pairs } \Pi_i \text{ and } \Gamma_i \text{ found by}$$

$$(12) \text{ is given by } \Pi_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^T, \Gamma_i = \begin{pmatrix} 0 & d_i/b_i \end{pmatrix},$$

$i = 1, 2, 3, 4, 5$. Let $R_i = 0.5$ for all the agents. Then the smallest allowable γ_i^* for each agent is found as $\gamma_1^* = 0.7160$, $\gamma_2^* = 0.9282$, $\gamma_3^* = 0.3724$, $\gamma_4^* = 2.1480$ and $\gamma_5^* = 1.1170$. Therefore, one can select $\gamma = 2.2$ for Problem 2. Using Theorem 4, the matrix gains become $K_1 = -1.4935$, $K_2 = -1.4935$, $K_3 = -1.4382$, $K_4 = -5.3358$ and $K_5 = -1.6142$. Using these gains, the outputs y_i are shown in Fig. 2(a), for the disturbance free case. One can see that all agents track the leader and part 1 of Problem 2 is verified. Then, disturbances ω_i are set as $\omega_1 = 0.5\sin(t)$, $\omega_2 = \sin(t)$, $\omega_3 = -\sin(t)$, $\omega_4 = 1.5\sin(t)$, $\omega_5 = 2\sin(t)$. The outputs y_i are shown in Fig. 2(b) and the global disturbance attenuation is shown in Fig. 3, from which one can see that (13) is satisfied.

Fig. 2. Outputs y_i (no disturbance) in 2(a); Outputs y_i (with disturbance) in 2(b)

Fig. 3. Disturbance Attenuation

6 Conclusion

$\mathcal{L}_2$ -gain synchronization of linear MAS subject to external disturbances is considered. A static OPFB control protocol is designed for homogeneous MAS whereas a dynamic OPFB control protocol is introduced for heterogeneous MAS. The problem can be cast into analyzing the associated decoupled systems where the control parameters are well selected using the LQR based optimal design.

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