Multi-Agent Zero-Sum Differential Graphical Games for Disturbance Rejection in Distributed Control

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Abstract

This paper addresses distributed optimal tracking control of multi-agent linear systems subject to external disturbances. The concept of differential game theory is utilized to formulate this distributed control problem into a multi-player zero-sum differential graphical game, which provides a new perspective on distributed tracking of multiple agents influenced by disturbances. In the presented differential graphical game, the dynamics and performance indices for each node depend on local neighbor information and disturbances. It is shown that the solution to the multi-agent differential graphical games in the presence of disturbances requires the solution to coupled Hamilton-Jacobi-Isaacs (HJI) equations. Multi-agent learning policy iteration (PI) algorithm is provided to find the solution to these coupled HJI equations and its convergence is proven. It is also shown that L_2 -bounded synchronization errors can be guaranteed using this technique. An online PI algorithm is given to solve the zero-sum game in real time. A simulation example is provided to show the effectiveness of the online approach.

Key words: Graphical games, Hamilton-Jacobi-Isaacs equations, L_2 -gain, External disturbances, Multi-agent system

1. Introduction

The distributed control of multi-agent systems has received much attention in the past years, due to its potential applications in a variety of engineering systems. A rich body of literature has been developed on distributed control methods for consensus and synchronization (Fax & Murray, 2004; Jadbabaie, Lin & Morse, 2003; Olfati-Saber & Murray, 2004; Qu, 2009; Ren & Beard, 2005, 2008; Ren, Beard, & Atkins, 2005; Tsitsiklis, 1984), and some good surveys can be found in (Lewis et al., 2014; Qu, 2009; Ren, Beard, & Atkins, 2005).

The objective in distributed control is to design for each agent a control protocol, depending only on local neighbor information, to guarantee synchronized behavior of all the agents by reaching agreement on certain quantities of interests. Although agent dynamics may be subject to external disturbances, most of the existing distributed control methods ignore these disturbances. However, to avoid performance degradation, it is necessary to consider the effect of disturbances in the distributed control problem formulation a priori.

Recently, disturbance attenuation has been taken into account in distributed control of multi-agent systems. Lin,

Jia, and Li (2008) studied the H_∞ consensus problem for first-order dynamical systems. Li, Duan & Huang (2009) transformed the problem of the disturbance rejection for multi-agent linear systems to the design of a set of H_∞ controllers for a set of independent systems. Liu and Jia (2010 and 2011) derived conditions in terms of linear matrix inequalities (LMIs) to ensure consensus of the multi-agent systems with a prescribed H_∞ level. Li, Duan, and Chen (2011) considered the distributed H_2 and H_∞ control problems for linear multi-agent systems. Wen, Duan, Li, and Chen (2012) addressed a finite L_2 -gain performance index for nonlinear multi-agent systems. Yang and Wang (2013) presented notions of L_p -gain and L_2 -gain consensus for multi-agent systems in the presence of disturbances.

Most of existing methods for multi-agent control in the presence of disturbances considered the leaderless consensus (or distributed regulation) problem, in which all nodes converge to a common value that cannot generally be controlled. On the other hand, the problem of distributed tracking (or leader-follower consensus problem), which is the problem of interest of this paper, requires that all nodes synchronize to a leader (Hong, Hu, & Gao, 2006; Li, Wang, & Chen, 2004; Ren, Moore, &

Chen, 2007; Wang, & Chen, 2002). Moreover, most of these methods derived conditions based on linear matrix inequalities (LMIs) to solve the problem (except for Wen et al. 2012; Yang & Wang, 2013). A new framework based on differential game theory was developed to achieve consensus of multi-agent systems (Vamvoudakis, Carrillo, & Hespanha, 2013) and synchronization of multi-wheeled mobile robots (Luy, Thanh, & Tri, 2013). Specifically, Vamvoudakis, Carrillo, and Hespanha (2013) presented a distributed algorithm, but it is limited to double integrator systems. Moreover, the convergence proof of the presented policy iteration algorithm and the L_2 -bounded synchronization error were not shown.

Over the last decade, multi-agent learning systems have been developed to create agents that learn from experience how to best interact with other agents (Busoniu, Babuska, & De Schutter, 2008; Chang, 2009; Lakshmanan & Farias, 2006; Littman, 2001; Vrancx, Verbeeck, & Nowe, 2008; Wheeler & Narendra, 1986). A significant part of the research on multi-agent learning concerns reinforcement learning techniques for finite state Markov decision processes. Convergence of each agent to the optimal response using Q-learning was shown on condition that all other agents converge to their optimal response (Littman, 2001). To the best of our knowledge, the multi-agent reinforcement learning for general continuous-time and continuous-state systems in the presence of disturbance has not yet been considered. Moreover, a rigorous proof of the convergence of the multi-agent learning methods to the optimal Nash equilibrium has not yet been provided in the general case.

The main contributions of the paper are as follows. First, a “Bounded L_2 -Gain Synchronization Problem” is formulated for multi-agent systems in the presence of disturbances in the agent dynamics. Second, the concept of differential game theory is utilized to formulate the distributed L_2 -Gain control problem into a multi-player zero-sum differential graphical game. Next, it is shown that the solution to this differential graphical game requires the solution to coupled Hamilton-Jacobi-Isaacs (HJI) equations. The Nash solution of the graphical game is investigated and the global synchronization error is shown to be L_2 -bounded if the graph has a spanning tree. Then, a multi-agent policy iteration algorithm is presented to find the solution to these HJI equations. A rigorous proof of the convergence of the proposed learning algorithm to the optimal Nash equilibrium is presented. This work extends the work of Vamvoudakis, Lewis, and Hudas (2012) to the cases in which the external disturbances cannot be ignored, on one hand, and the work of Vamvoudakis, Carrillo, and Hespanha (2013) to the distributed tracking problem with general linear systems, on the other hand.

This paper is organized as follows. The next section provides mathematical background and the problem formulation for distributed L_2 -gain control of multi-agent systems. This problem is then formulated into a multi-player zero-sum differential graphical game in Section 3. The Nash solution to this graphical game is presented in Section 4. Sections 5 and 6 present policy iteration

algorithms and their implementation, respectively, for finding the Nash solution to the presented game. Sections 7 and 8 present simulation results and conclusion, respectively.

2. Preliminaries and Problem Formulation

In this section, a review of communication graphs is given and the problem of synchronization of multi-agent systems subject to external disturbances is formulated.

2.1. Mathematical Background

A *directed graph* G consists of a pair (V, E) , where $V = \{\alpha_1, \dots, \alpha_N\}$ is a finite nonempty set of nodes and $E \subseteq V \times V$ is a set of ordered pairs of nodes, called edges. $E = [e_{ij}]$ is called the adjacency matrix with $e_{ij} > 0$ if $(\alpha_j, \alpha_i) \in E$ and $e_{ij} = 0$ otherwise. Note that diagonal elements $e_{ii} = 0$. The set of nodes α_j with edges incoming to node α_i is called the neighbors of node i , namely $N_i = \{\alpha_j : (\alpha_j, \alpha_i) \in E\}$. The graph Laplacian matrix is defined as $L = D - E$, which has all row sums equal to zero. $D = \text{diag}(d_i)$ is called the in-degree matrix, where $d_i = \sum_{j \in N_i} e_{ij}$ is the weighted in-degree of node i .

Definition 1. (Lewis et al., 2014) A (directed) tree is a connected digraph where every node except one, called the root, has in-degree equal to one. A graph is said to have a spanning tree if a subset of the edges forms a directed tree.

Throughout the paper, $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$ are denoted as the maximum and minimum singular values of the matrix A , respectively.

2.2. Problem Formulation

Consider the communication graph $G = (V, E)$ having N agents, each with dynamics given by

$$\dot{x}_i = Ax_i + B_i u_i + D_i v_i \quad (1)$$

where $x_i(t) \in R^n$, $u_i(t) \in R^{m_i}$ and $v_i(t) \in R^{q_i}$ are the state, control input and external disturbance of node i , respectively. The state of the control or leader node is $x_0(t) \in R^n$ and it is assumed to satisfy the dynamics

$$\dot{x}_0 = Ax_0 \quad (2)$$

Standard Synchronization Problem. Design $u_i(t)$ in (1) so that $\|x_i(t) - x_0(t)\| \rightarrow 0, \forall i$ when $v_i(t) = 0$.

For each node i , the local neighborhood tracking error $\delta_i \in R^n$ is defined as (Khoo, Xie, & Man, 2009)

$$\delta_i = \sum_{j \in N_i} e_{ij} (x_i - x_j) + g_i (x_i - x_0) \quad (3)$$

where $g_i \geq 0$ is called the pinning gain and $g_i > 0$ for at least one root node i (Li, Wang, & Chen, 2004).

The overall tracking error vector for all nodes is given by

$$\delta = ((L + G) \otimes I_n)(x - \underline{x}_0) = ((L + G) \otimes I_n)\zeta \quad (4)$$

where $x = [x_1^T \ x_2^T \ \dots \ x_N^T]^T$, $\delta = [\delta_1^T \ \delta_2^T \ \dots \ \delta_N^T]^T$ are global node state vector and the global tracking error vector, and $\underline{x}_0 = \underline{I}x_0 \in R^{nN}$, with $\underline{I} = \mathbf{1}_N \otimes I_n \in R^{nN \times n}$, I_n the $n \times n$ identity matrix and $\mathbf{1}_N$ the N -vector of ones. The

Kronecker product is denoted by $\otimes$. The pinning gain matrix $G \in R^{N \times N}$ is a diagonal matrix with diagonal entries equal to the pinning gains g_i . The (global) synchronization error is

$$\zeta = (x - x_0) \in R^{nN} \quad (5)$$

The following Lemma shows that the small local neighborhood synchronization error implies small global synchronization error.

Lemma 1. (Khoo, Xie, & Man, 2009) Let $(L + G)$ be non-singular. Then the synchronization error is bounded by

$$\|\zeta\| \leq \|\delta\| / \underline{\sigma}(L + G) \quad (6)$$

Remark 1. The matrix $(L + G)$ is non-singular if the graph has a spanning tree and $g_i \neq 0$ for a root node i (Khoo, Xie, & Man, 2009).

The local neighborhood tracking error dynamics for node i is defined as

$$\dot{\delta}_i = \sum_{j \in N_i} e_{ij} (\dot{x}_i - \dot{x}_j) + g_i (\dot{x}_i - \dot{x}_0) \quad (7)$$

which by using (1) and (2) becomes

$$\begin{aligned} \dot{\delta}_i &= A\delta_i + (d_i + g_i)B_i u_i - \sum_{j \in N_i} e_{ij} B_j u_j + (d_i + g_i)D_i v_i \\ &\quad - \sum_{j \in N_i} e_{ij} D_j v_j \end{aligned} \quad (8)$$

with $\delta_i \in R^n$, $u_i \in R^{m_i}$, $v_i \in R^{q_i}$, $\forall i$. This is a dynamical system with multiple control inputs and disturbance inputs from node i and all of her neighbors.

The objective here is to design a distributed control protocol to make the tracking synchronization error (3) L_2 -bounded when $v_i(t) \neq 0$.

Bounded L_2 -Gain Synchronization Problem. Consider the systems (8) with measured outputs $y_i = H_{ii} \delta_i$ (Note that H_{ii} must be left invertible and hence δ_i can be directly measured), disturbances $\omega_i(t) = [v_i^T(t) \ v_{-i}^T(t)]^T$ and performance outputs $z_i(t) = [\delta_i^T(t) \ u_i^T(t) \ u_{-i}^T(t)]^T$ with $v_{-i}(t) = \{v_j : j \in N_i\}$ and $u_{-i}(t) = \{u_j : j \in N_i\}$. It is desired to select the control $u_i(t)$ to solve the standard synchronization problem when $v_i(t) = 0$, and also to satisfy the following bounded L_2 -gain condition for a given $\gamma > 0$ for all agents when $v_i(t) \neq 0$,

$$\begin{aligned} \int_0^T \|z_i(t)\|^2 dt &= \int_0^T (\delta_i^T Q_{ii} \delta_i + u_i^T R_{ii} u_i + \sum_{j \in N_i} u_j^T R_{ij} u_j) dt \\ &\leq \gamma^2 \int_0^T \|\omega_i(t)\|^2 dt + \beta(\delta_i(0)) \\ &= \gamma^2 \int_0^T (v_i^T T_{ii} v_i + \sum_{j \in N_i} v_j^T T_{ij} v_j) dt + \beta(\delta_i(0)) \end{aligned} \quad (9)$$

for some bounded function β such that $\beta(0) = 0$ (Aliyu, 2011), with $Q_{ii} > 0$, $R_{ii} > 0$, $R_{ij} \geq 0$, $T_{ii} > 0$, $T_{ij} \geq 0$. Let γ^* be the minimum value of γ for which the disturbance attenuation condition (9) is satisfied.

3. Multi-Player Zero-Sum Differential Graphical Game

Define the following local performance index for each agent.

$$\begin{aligned} J_i(\delta_i(0), u_i, u_{-i}, v_i, v_{-i}) &= \frac{1}{2} \int_0^\infty (\delta_i^T Q_{ii} \delta_i + u_i^T R_{ii} u_i \\ &\quad + \sum_{j \in N_i} u_j^T R_{ij} u_j - \gamma^2 v_i^T T_{ii} v_i - \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j) dt \end{aligned} \quad (10)$$

3.1. Zero-Sum Differential Graphical Games

It will be shown in section 4 that the solvability of the bounded L_2 -gain synchronization problem is equivalent to the solvability of the following multi-player zero-sum differential game

$$V_i(\delta_i(0)) = \min_{u_i} \max_{v_i} J_i(\delta_i(0), u_i, u_{-i}, v_i, v_{-i}) \quad (11)$$

where J_i is defined in (10). The control player seeks to minimize the value and the disturbance player to maximize it. This game has a unique solution if a game theoretic saddle point (u_i^*, v_i^*) exists. That is, if (Basar & Bernard, 2008; Lewis, Vrabie, & Syrmos, 2012)

$$\begin{aligned} V_i^*(\delta_i(0)) &= \min_{u_i} \max_{v_i} J_i(\delta_i(0), u_i, u_{-i}^*, v_i, v_{-i}^*) \\ &= \max_{v_i} \min_{u_i} J_i(\delta_i(0), u_i, u_{-i}^*, v_i, v_{-i}^*) \end{aligned} \quad (12)$$

The associated value V_i^* is called the value of the game. This is equivalent to the Nash equilibrium condition

$$J_i(u_i^*, u_{-i}^*, v_i, v_{-i}^*) \leq J_i(u_i^*, u_{-i}^*, v_i^*, v_{-i}^*) \leq J_i(u_i, u_{-i}^*, v_i^*, v_{-i}^*) \quad (13)$$

holding for all policies u_i, v_i . Interpreting the control inputs $u_i, \forall i$, and the disturbance inputs $v_i, \forall i$ as policies in terms of δ_i , the value function for node i is

$$\begin{aligned} V_i(\delta_i(t)) &= \frac{1}{2} \int_t^\infty (\delta_i^T Q_{ii} \delta_i + u_i^T R_{ii} u_i + \sum_{j \in N_i} u_j^T R_{ij} u_j \\ &\quad - \gamma^2 v_i^T T_{ii} v_i - \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j) dt \end{aligned} \quad (14)$$

Using Leibniz's formula, a differential equivalent to (14) is given by the Bellman equation as

$$\begin{aligned} H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}, u_i, u_{-i}, v_i, v_{-i}) &\equiv \frac{\partial V_i}{\partial \delta_i} (A\delta_i + (d_i + g_i)B_i u_i \\ &\quad - \sum_{j \in N_i} e_{ij} B_j u_j + (d_i + g_i)D_i v_i - \sum_{j \in N_i} e_{ij} D_j v_j) + \frac{1}{2} \delta_i^T Q_{ii} \delta_i \\ &\quad + \frac{1}{2} u_i^T R_{ii} u_i + \frac{1}{2} \sum_{j \in N_i} u_j^T R_{ij} u_j - \frac{1}{2} \gamma^2 v_i^T T_{ii} v_i - \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j = 0 \end{aligned} \quad (15)$$

with the boundary condition $V_i(0) = 0$. The stationarity conditions give (Lewis, Vrabie, & Syrmos, 2012)

$$0 = \frac{\partial H_i}{\partial u_i} \Rightarrow u_i = -(d_i + g_i)R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} \quad (16)$$

$$0 = \frac{\partial H_i}{\partial v_i} \Rightarrow v_i = \frac{1}{\gamma^2} (d_i + g_i)T_{ii}^{-1} D_i^T \frac{\partial V_i}{\partial \delta_i} \quad (17)$$

Remark 2. The game defined by (10), (11) is in fact a zero-sum game with respect to u_i and v_i for each agent i , with outcome coupled through the neighboring relations expressed by the sets N_i .

3.2. Coupled Hamilton-Jacobi-Isaacs Equations

Substituting control policies (16) and disturbance policies (17) into (15) yields the coupled HJI equations

$$\begin{aligned} & \frac{\partial V_i^T}{\partial \delta_i} A_i^c + \frac{1}{2} \delta_i^T Q_{ii} \delta_i + \frac{1}{2} (d_i + g_i)^2 \frac{\partial V_i^T}{\partial \delta_i} B_i R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} \\ & + \frac{1}{2} \sum_{j \in N_i} (d_j + g_j)^2 \frac{\partial V_j^T}{\partial \delta_j} B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \frac{\partial V_j}{\partial \delta_j} \\ & - \frac{1}{2\gamma^2} (d_i + g_i)^2 \frac{\partial V_i^T}{\partial \delta_i} D_i T_{ii}^{-1} D_i^T \frac{\partial V_i}{\partial \delta_i} \\ & - \frac{1}{2\gamma^2} \sum_{j \in N_i} (d_j + g_j)^2 \frac{\partial V_j^T}{\partial \delta_j} D_j T_{jj}^{-1} T_{ij} T_{jj}^{-1} D_j^T \frac{\partial V_j}{\partial \delta_j} = 0 \end{aligned} \quad (18)$$

with boundary condition $V_i(0) = 0$, where

$$\begin{aligned} A_i^c &= A \delta_i - (d_i + g_i)^2 B_i R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} + \sum_{j \in N_i} e_{ij} (d_j + g_j) \times \\ & B_j R_{jj}^{-1} B_j^T \frac{\partial V_j}{\partial \delta_j} + \frac{1}{\gamma^2} (d_i + g_i)^2 D_i T_{ii}^{-1} D_i^T \frac{\partial V_i}{\partial \delta_i} \\ & - \frac{1}{\gamma^2} \sum_{j \in N_i} e_{ij} (d_j + g_j) D_j T_{jj}^{-1} D_j^T \frac{\partial V_j}{\partial \delta_j} \end{aligned}$$

is the closed-loop system corresponding to agent i . For a given solution V_i , define $u_i^* = u_i(V_i)$, $u_{-i}^* = u_{-i}(V_i)$, $v_i^* = v_i(V_i)$ and $v_{-i}^* = v_{-i}(V_i)$ in terms of V_i , then (18) can be written as

$$H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}, u_i^*, u_{-i}^*, v_i^*, v_{-i}^*) = 0, V_i(0) = 0 \quad (19)$$

Lemma 2. Assume that the neighboring agents' policies are already optimal. Then, for every u_i, v_i , the following holds

$$\begin{aligned} H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}, u_i, u_{-i}^*, v_i, v_{-i}^*) &= \frac{1}{2} (u_i - u_i^*)^T R_{ii} (u_i - u_i^*) \\ & - \frac{1}{2} \gamma^2 (v_i - v_i^*)^T T_{ii} (v_i - v_i^*) \end{aligned} \quad (20)$$

Proof. Completing the squares in (15) gives (20). ■

4. L₂-Bounded Synchronization Error and Nash Solution to the Graphical Games

4.1. L₂-Bounded Synchronization Error

In this subsection, we want to find the control inputs $u_i(t)$ such that, for all disturbances $v_i(t) \in L_2[0, \infty)$ the condition (9) holds for a prescribed $\gamma > 0$. The following result provides a solution to the bounded L₂-gain problem in terms of a solution to the HJI equation (18).

Theorem 1. Solution to Bounded L₂-gain Problem. Let $\gamma \geq \gamma^*$. Suppose $V_i^* > 0$, $i \in N$ is a smooth positive definite solution to the HJI equation (18). Assume that the neighboring agents' policies are already optimal. Then, the equilibrium point of the closed-loop system

$$\dot{\delta}_i = A \delta_i + (d_i + g_i) B_i u_i^* - \sum_{j \in N_i} e_{ij} B_j u_j^* \quad (21)$$

is asymptotically stable with control inputs $u_i^* = u_i(V_i^*)$, given by (16) in terms of V_i^* . Moreover, for these choices of control inputs, the bounded L₂-gain condition (9) holds for all disturbances $v_i(t) \in L_2[0, \infty)$, where $Q_{ii} = H_{ii}^T H_{ii}$.

Proof. Suppose now that V_i^* satisfies the coupled HJI equation (18) and set $u_{-i} = u_{-i}^*, v_{-i} = v_{-i}^*$. Then, using (20) in the Hamiltonian function (15) in terms of V_i^* , one has

$$\begin{aligned} H_i(\delta_i, \frac{\partial V_i^*}{\partial \delta_i}, u_i, u_{-i}, v_i, v_{-i}) &= \frac{dV_i^*}{dt} + \frac{1}{2} \delta_i^T Q_{ii} \delta_i + \frac{1}{2} u_i^T R_{ii} u_i \\ & + \frac{1}{2} \sum_{j \in N_i} u_j^T R_{ij} u_j - \frac{1}{2} \gamma^2 v_i^T T_{ii} v_i - \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j \\ & = \frac{1}{2} (u_i - u_i^*)^T R_{ii} (u_i - u_i^*) - \frac{1}{2} \gamma^2 (v_i - v_i^*)^T T_{ii} (v_i - v_i^*) \end{aligned}$$

Selecting $u_i = u_i^*$ yields

$$\begin{aligned} \frac{dV_i^*}{dt} &+ \frac{1}{2} \delta_i^T Q_{ii} \delta_i + \frac{1}{2} u_i^T R_{ii} u_i + \frac{1}{2} \sum_{j \in N_i} u_j^T R_{ij} u_j \\ &- \frac{1}{2} \gamma^2 v_i^T T_{ii} v_i - \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j \leq 0 \end{aligned} \quad (22)$$

for all v_i, v_j . To show asymptotic stability of the closed-loop system (21), set $v_i = v_j = 0$ in the above equation. Then, one has

$$\frac{dV_i^*}{dt} \leq -\frac{1}{2} (\delta_i^T Q_{ii} \delta_i + u_i^T R_{ii} u_i + \sum_{j \in N_i} u_j^T R_{ij} u_j) \leq -\frac{1}{2} \delta_i^T Q_{ii} \delta_i$$

Therefore, the equilibrium point of the system is asymptotically stable using the Lyapunov Theorem. To show that the condition (9) is satisfied, integrating (22) yields,

$$\begin{aligned} V_i^*(\delta_i(t)) - V_i^*(\delta_i(0)) &+ \frac{1}{2} \int_0^t (\delta_i^T Q_{ii} \delta_i + u_i^T R_{ii} u_i \\ &+ \sum_{j \in N_i} u_j^T R_{ij} u_j - \gamma^2 v_i^T T_{ii} v_i - \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j) dt \leq 0 \end{aligned}$$

Noting that $V_i^*(\delta_i(t)) > 0$, one has

$$\begin{aligned} & \int_0^t (\delta_i^T Q_{ii} \delta_i + u_i^T R_{ii} u_i + \sum_{j \in N_i} u_j^T R_{ij} u_j) dt \\ & \leq \gamma^2 \int_0^t (v_i^T T_{ii} v_i + \sum_{j \in N_i} v_j^T T_{ij} v_j) dt + V_i^*(\delta_i(0)) \end{aligned}$$

Therefore, the L₂ gain is bounded by γ and (9) is satisfied. ■

Corollary 1. Assume that the assumptions made in the statement of Theorem 1 are satisfied. Then, the global synchronization error ζ is L₂-bounded, if the graph has a spanning tree.

Proof. By Theorem 1 and Lemma 1, one can conclude that the global synchronization error is L₂-bounded, if the graph has a spanning tree. ■

4.2. Nash Solution to the Graphical Game

The following result shows that under certain conditions the HJI solution satisfies the Nash condition (13) and so solves the zero-sum game.

Theorem 2. Solution to multi-player zero-sum game. Let $\gamma \geq \gamma^*$. Assume that the game (11) has a finite value and the neighboring agents' policies are already optimal. Suppose $V_i^* > 0$, $i \in N$ is a smooth positive definite solution to the HJI equation (18) such that the equilibrium point of the closed-loop system

$$\begin{aligned} \dot{\delta}_i = & A\delta_i + (d_i + g_i)B_i u_i^* - \sum_{j \in N_i} e_{ij} B_j u_j^* \\ & + (d_i + g_i)D_i v_i^* - \sum_{j \in N_i} e_{ij} D_j v_j^* \end{aligned} \quad (23)$$

is asymptotically stable. Then, the Nash condition (13) is satisfied for control policy $u_i^* = u_i(V_i^*)$ and disturbance policy $v_i^* = v_i(V_i^*)$ as (16) and (17), given in terms of V_i^* . Moreover, the value of the game is given by the HJI solution $V_i^*(\delta_i(0))$.

Proof. For any smooth functions $V_i(\delta_i)$, $i \in N$, one can write (10) as

$$\begin{aligned} & J_i(\delta_i(0), u_i, u_{-i}, v_i, v_{-i}) \\ = & \frac{1}{2} \int_0^\infty (\delta_i^T Q_{ii} \delta_i + u_i^T R_{ii} u_i + \sum_{j \in N_i} u_j^T R_{ij} u_j - \gamma^2 v_i^T T_{ii} v_i \\ & - \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j) dt + V_i(\delta_i(0)) - V_i(\delta_i(\infty)) + \int_0^\infty \frac{\partial V_i^T}{\partial \delta_i} \\ & (A\delta_i + (d_i + g_i)B_i u_i - \sum_{j \in N_i} e_{ij} B_j u_j + (d_i + g_i)D_i v_i \\ & - \sum_{j \in N_i} e_{ij} D_j v_j) dt \end{aligned} \quad (24)$$

Now let V_i^* satisfies (18) and $u_i^*, u_{-i}^*, v_i^*, v_{-i}^*$ be the optimal policies given by (16) and (17). Completing the squares yields as

$$\begin{aligned} & J_i(\delta_i(0), u_i, u_{-i}, v_i, v_{-i}) \\ = & V_i^*(\delta_i(0)) - V_i^*(\delta_i(\infty)) + \int_0^\infty (\frac{1}{2} \sum_{j \in N_i} (u_j - u_j^*)^T R_{ij} (u_j - u_j^*) \\ & + \frac{1}{2} (u_i - u_i^*)^T R_{ii} (u_i - u_i^*) - \frac{\partial V_i^T}{\partial \delta_i} \sum_{j \in N_i} e_{ij} B_j (u_j - u_j^*) \\ & + \sum_{j \in N_i} u_j^{*T} R_{ij} (u_j - u_j^*) - \frac{1}{2} \gamma^2 \sum_{j \in N_i} (v_j - v_j^*)^T T_{ij} (v_j - v_j^*) \\ & - \frac{1}{2} \gamma^2 (v_i - v_i^*)^T T_{ii} (v_i - v_i^*) - \frac{\partial V_i^T}{\partial \delta_i} \sum_{j \in N_i} e_{ij} D_j (v_j - v_j^*) \\ & - \gamma^2 \sum_{j \in N_i} v_j^{*T} T_{ij} (v_j - v_j^*)) dt \end{aligned} \quad (25)$$

Since the equilibrium point of the closed-loop system (23) is asymptotically stable, one has $V_i^*(\delta_i(\infty)) = 0$.

Assuming $u_j = u_j^*, v_j = v_j^*, \forall j \in N_i$, then we obtain

$$\begin{aligned} J_i(\delta_i(0), u_i, u_{-i}^*, v_i, v_{-i}^*) = & V_i^*(\delta_i(0)) + \int_0^\infty (\frac{1}{2} (u_i - u_i^*)^T \\ & R_{ii} (u_i - u_i^*) - \frac{1}{2} \gamma^2 (v_i - v_i^*)^T T_{ii} (v_i - v_i^*)) dt \end{aligned} \quad (26)$$

Due to its quadratic form, (26) shows the Nash condition (13) is satisfied. In fact, if the strategy of the disturbance input (the control input) remains unchanged, the control input (the disturbance input) does not gain anything based on (26). Moreover, setting $u_i = u_i^*, v_i = v_i^*$ in (26) leads to

$$J_i^*(\delta_i(0), u_i^*, u_{-i}^*, v_i^*, v_{-i}^*) = V_i^*(\delta_i(0)) \quad (27)$$

which gives the optimal game value. This completes the proof. ■

5. Multi-agent Policy Iteration Algorithm to solve HJI equations

5.1. Best response

Definition 2. Given fixed neighbor control policies u_{-i} and disturbance policies v_{-i} , assume that there exist stabilizing control policy u_i and disturbance policy v_i . Then, agent i 's best control and disturbance response to fixed policies u_{-i} and v_{-i} of her neighbors are the pair of policies u_i^* and v_i^* such that

$$J_i(u_i^*, u_{-i}, v_i, v_{-i}) \leq J_i(u_i^*, u_{-i}, v_i^*, v_{-i}) \leq J_i(u_i, u_{-i}, v_i^*, v_{-i}) \quad (28)$$

Define the best response HJI equation with control $u_i = u_i^*$ given by (16) and disturbances $v_i = v_i^*$ given by (17) and arbitrary policies u_{-i}, v_{-i} as

$$\begin{aligned} & \frac{\partial V_i^T}{\partial \delta_i} A_i^c + \frac{1}{2} \delta_i^T Q_{ii} \delta_i + \frac{1}{2} (d_i + g_i)^2 \frac{\partial V_i^T}{\partial \delta_i} B_i R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} \\ & + \frac{1}{2} \sum_{j \in N_i} u_j^T R_{ij} u_j - \frac{1}{2\gamma^2} (d_i + g_i)^2 \frac{\partial V_i^T}{\partial \delta_i} D_i T_{ii}^{-1} D_i^T \frac{\partial V_i}{\partial \delta_i} \\ & - \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j = 0 \end{aligned} \quad (29)$$

with boundary condition $V_i(0) = 0$, where $A_i^c, \forall i$ is the closed-loop system.

5.2. Policy iteration to solve the HJI equations

In this subsection, a policy iteration (PI) algorithm is presented to solve the best response HJI equations (29) and the HJI equations (18). The PI technique has two loops, i.e. an inner loop (policy evaluation) and an outer loop (policy improvement). In the inner loop, the control policies are fixed and the disturbances are updated using (17) by solving the Bellman equation (15). In the outer loop, improved control policies are found using (16). This gives the following PI algorithm.

Algorithm 1: Multi-agent Learning Policy Iteration for Solving HJI equations

Let $u_i^0, \forall i = 1, \dots, N$ be any stabilizing initial policy.

For $k = 0, 1, \dots$, given $u_i^k, \forall i = 1, \dots, N$.
repeat at each step k
for $l = 0, 1, \dots$, set $v_i^0 = 0$, (Policy evaluation)
repeat at each step l
solve for $V_i^{(k,l)}(\delta_i)$ using the zero-sum Bellman equation

$$\begin{aligned} & \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^k - \sum_{j \in N_i} e_{ij} B_j u_j \\ & + (d_i + g_i)D_i v_i^l - \sum_{j \in N_i} e_{ij} D_j v_j) + \frac{1}{2} \delta_i^T Q_i \delta_i \\ & + \frac{1}{2} u_i^{kT} R_i u_i^k + \frac{1}{2} \sum_{j \in N_i} u_j^T R_j u_j - \frac{1}{2} \gamma^2 v_i^{lT} T_{ii} v_i^l \\ & - \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j = 0 \quad \forall i = 1, \dots, N \end{aligned} \quad (30)$$

update the disturbances v_i^{l+1} according to

$$v_i^{l+1} = \frac{1}{\gamma^2} (d_i + g_i) T_{ii}^{-1} D_i^T \frac{\partial V_i^{(k,l)}}{\partial \delta_i} \quad \forall i = 1, \dots, N \quad (31)$$

until convergence, set $V_i^{(k+1,l)} = V_i^{(k,l)}$
end for
update the control policy (Policy improvement) using

$$u_i^{k+1} = -(d_i + g_i) R_i^{-1} B_i^T \frac{\partial V_i^{(k,l)}}{\partial \delta_i} \quad \forall i = 1, \dots, N \quad (32)$$

until convergence
end for

The following two theorems prove the convergence of the policy iteration Algorithm 1 for the best response HJI equation (29) and the coupled HJI equation (18), respectively. In Theorem 3, only agent i updates her policy and in Theorem 4, all the agents update their policies simultaneously.

Theorem 3. Assume that the policies in the neighborhood are fixed and that there exist a stabilizing control policy and disturbance policy with $\gamma \geq \gamma^*$. Assume that agent i performs Algorithm 1. Then the algorithm converges to the best response u_i and v_i to policies u_{-i} and v_{-i} of the neighbors, and the value converges to the optimal game value $V_i^{(k,l)} \rightarrow V_i^*$, which V_i^* is the solution to the best response HJI equation (29).

Proof. Firstly, we prove the convergence of the inner-loop. By integrating $\dot{V}_i^{(k,l)} - \dot{V}_i^{(k,l+1)}$ along the solution of the error system

$$\begin{aligned} \dot{\delta}_i &= A\delta_i + (d_i + g_i)B_i u_i^k - \sum_{j \in N_i} e_{ij} B_j u_j \\ &+ (d_i + g_i)D_i v_i^{l+1} - \sum_{j \in N_i} e_{ij} D_j v_j \end{aligned} \quad (33)$$

and considering that only agent i updates her policy, yields

$$\begin{aligned} V_i^{(k,l)} - V_i^{(k,l+1)} &= \int_t^\infty \left(\left(\frac{\partial V_i^{(k,l+1)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^k \right. \\ &- \sum_{j \in N_i} e_{ij} B_j u_j + (d_i + g_i)D_i v_i^{l+1} - \sum_{j \in N_i} e_{ij} D_j v_j) \\ &- \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^k - \sum_{j \in N_i} e_{ij} B_j u_j \\ &+ (d_i + g_i)D_i v_i^{l+1} - \sum_{j \in N_i} e_{ij} D_j v_j) \Big) dt \end{aligned} \quad (34)$$

Using (30) for the first and second terms of the right-hand side the above equation one has

$$\begin{aligned} V_i^{(k,l)} - V_i^{(k,l+1)} &= \int_t^\infty \left(\frac{1}{2} \gamma^2 v_i^{(l+1)T} T_{ii} v_i^{(l+1)} - \frac{1}{2} \gamma^2 v_i^{lT} T_{ii} v_i^l \right. \\ &+ \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i)D_i v_i^l - \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i)D_i v_i^{l+1} \Big) dt \end{aligned} \quad (35)$$

Then, substituting (31) into (35) and doing some manipulations gives

$$V_i^{(k,l)} - V_i^{(k,l+1)} = - \int_t^\infty \left(\frac{1}{2} \gamma^2 (v_i^{(l+1)} - v_i^l)^T T_{ii} (v_i^{(l+1)} - v_i^l) \right) dt \leq 0 \quad (36)$$

Therefore, one has $V_i^{(k,l)} \leq V_i^{(k,l+1)}$. Defining the sequence $a_l = V_i^{(k,l)}$, the inequality $V_i^{(k,l)} \leq V_i^{(k,l+1)}$ shows that $a_l - a_{l+1} \leq 0 \Leftrightarrow a_{l+1} \geq a_l$ and so $\{a_l\}$ is a monotonically increasing sequence. Additionally, $\{a_l\}$ is bounded from above by the value of the corresponding storage function. Therefore, the sequence $\{a_l\}$ converges to a limit. Note that the equality holds if the sequence $\{a_l\}$ has converged. To show this, let n be the first time for which $a_{n-1} = a_n$, then one has $v_i^{(n)} = v_i^{(n+1)}$ based on (31). Moreover, $a_n = a_{n+1}$ according to (36). This, on the other hand, based on (31) implies that $v_i^{(n+1)} = v_i^{(n+2)}$ and consequently $a_{n+1} = a_{n+2}$. Similarly, one can show that $a_{n+i} = a_{n+i+1} \forall i \geq 0$ and thus the convergence is achieved. This completes the proof of convergence for the inner-loop.

To prove the convergence of the outer-loop, integrating $\dot{V}_i^{(k,l)} - \dot{V}_i^{(k+1,l)}$ along the solution of the error system gives

$$\begin{aligned} \dot{\delta}_i &= A\delta_i + (d_i + g_i)B_i u_i^{k+1} - \sum_{j \in N_i} e_{ij} B_j u_j \\ &+ (d_i + g_i)D_i v_i^l - \sum_{j \in N_i} e_{ij} D_j v_j \end{aligned} \quad (37)$$

and considering that only agent i updates her policy, one has

$$\begin{aligned} V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(\left(\frac{\partial V_i^{(k+1,l)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^{k+1} \right. \\ &- \sum_{j \in N_i} e_{ij} B_j u_j + (d_i + g_i)D_i v_i^l - \sum_{j \in N_i} e_{ij} D_j v_j) \end{aligned}$$

$$\begin{aligned} & - \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T \left(A\delta_i + (d_i + g_i)B_i u_i^{k+1} - \sum_{j \in N_i} e_{ij} B_j u_j \right. \\ & \left. + (d_i + g_i)D_i v_i^l - \sum_{j \in N_i} e_{ij} D_j v_j^l \right) dt \end{aligned} \quad (38)$$

Using (30) for the first and second terms of the right-hand side of the above equation one has

$$\begin{aligned} V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(\frac{1}{2} u_i^{kT} R_{ii} u_i^k - \frac{1}{2} u_i^{(k+1)T} R_{ii} u_i^{(k+1)} + \right. \\ & \left. \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i) B_i u_i^k - \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i) B_i u_i^{k+1} \right) dt \end{aligned} \quad (39)$$

Then, substituting (32) into (39) and doing some manipulations gives

$$V_i^{(k,l)} - V_i^{(k+1,l)} = \int_t^\infty \left(\frac{1}{2} (u_i^{(k+1)} - u_i^k)^T R_{ii} (u_i^{(k+1)} - u_i^k) \right) dt \geq 0 \quad (40)$$

Therefore, $V_i^{(k,l)} \geq V_i^{(k+1,l)}$. Defining the sequence $a_k = V_i^{(k,l)}$, the inequality $V_i^{(k,l)} \geq V_i^{(k+1,l)}$ shows that $a_k \geq a_{k+1}$. Therefore, the sequence $\{a_k\}$ is monotonically decreasing and bounded from below, and so it converges to a limit. Note that the equality holds if the sequence $\{a_k\}$ has converged. To show this, Let m be the first time for which $a_{m-1} = a_m$, then one has $u_i^{(m)} = u_i^{(m+1)}$ based on (32). Moreover, $a_m = a_{m+1}$ according to (40). This, on the other hand, based on (32) implies that $u_i^{(m+1)} = u_i^{(m+2)}$ and consequently $a_{m+1} = a_{m+2}$. Similarly, one can show that $a_{m+i} = a_{m+i+1}$, $\forall i \geq 0$ and thus the convergence is achieved. This completes the proof of convergence for the inner-loop.

Consequently, according to the above results and the principle of optimality, one can conclude that the algorithm converges to V_i^* , the solution to the best response HJI equation (29). ■

The next theorem proves the convergence of Algorithm 1 for the case where all agents update their policies simultaneously. Define the relative weights $\rho_{ij} = \bar{\sigma}(R_{jj}^{-1} R_{ij})$, and $\beta_{ij} = \bar{\sigma}(T_{jj}^{-1} T_{ij})$.

Theorem 4. Assume that the policies in the neighborhood are updated and that there exist a stabilizing control policy and disturbance policy with $\gamma \geq \gamma^*$. Assume that agent i performs Algorithm 1. Then, for small weights e_{ij} , and relative weights β_{ij} , ρ_{ij} , the policies u_i, u_j, v_i, v_j converge to the Nash equilibrium, and the game values converge to $V_i^{(k,l)} \rightarrow V_i^*$, which V_i^* are the solutions to the HJI equations (18).

Proof. Firstly, we prove the convergence of the inner-loop. By integrating $\dot{V}_i^{(k,l)} - \dot{V}_i^{(k+1,l)}$ along the solution of the error system

$$\begin{aligned} \dot{\delta}_i &= A\delta_i + (d_i + g_i)B_i u_i^k - \sum_{j \in N_i} e_{ij} B_j u_j^k \\ &+ (d_i + g_i)D_i v_i^{l+1} - \sum_{j \in N_i} e_{ij} D_j v_j^l \end{aligned} \quad (41)$$

and considering that all agents update their policies, one has

$$\begin{aligned} V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(\left(\frac{\partial V_i^{(k,l+1)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^k \right. \\ &- \sum_{j \in N_i} e_{ij} B_j u_j^k + (d_i + g_i)D_i v_i^{l+1} - \sum_{j \in N_i} e_{ij} D_j v_j^l) \\ &- \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^k - \sum_{j \in N_i} e_{ij} B_j u_j^k \\ &+ (d_i + g_i)D_i v_i^{l+1} - \sum_{j \in N_i} e_{ij} D_j v_j^l) \Big) dt \end{aligned} \quad (42)$$

Using (30) for the first and second the above equation yields

$$\begin{aligned} V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(\frac{1}{2} \gamma^2 v_i^{(l+1)T} T_{ii} v_i^{(l+1)} + \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^{(l+1)T} T_{ij} v_j^{(l+1)} \right. \\ &+ \left(\frac{\partial V_i^{(k,l+1)}}{\partial \delta_i} \right)^T \sum_{j \in N_i} e_{ij} D_j v_j^{l+1} - \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T \sum_{j \in N_i} e_{ij} D_j v_j^l \\ &- \frac{1}{2} \gamma^2 v_i^{lT} T_{ii} v_i^l - \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^{lT} T_{ij} v_j^l + \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i) D_i v_i^l \\ &- \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i) D_i v_i^{l+1} \Big) dt \end{aligned} \quad (43)$$

Then, substituting (31) into (43) and doing some manipulations gives

$$\begin{aligned} V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(-\frac{1}{2} \gamma^2 (v_i^{l+1} - v_i^l)^T T_{ii} (v_i^{l+1} - v_i^l) \right. \\ &- \frac{1}{2} \gamma^2 \sum_{j \in N_i} (v_j^{l+1} - v_j^l)^T T_{ij} (v_j^{l+1} - v_j^l) + \gamma^2 \sum_{j \in N_i} v_j^{(l+1)T} T_{ij} \times \\ &\left. (v_j^{l+1} - v_j^l) + \sum_{j \in N_i} \left(\frac{\partial V_i^{(k,l+1)}}{\partial \delta_i} \right)^T e_{ij} D_j (v_j^{l+1} - v_j^l) \right) dt \end{aligned} \quad (44)$$

A sufficient condition for $V_i^{(k,l)} \leq V_i^{(k+1,l)}$ is

$$\begin{aligned} & \frac{1}{2} \gamma^2 \underline{\sigma}(T_{ii}) \|\Delta v_i\| > (d_i + g_i) \beta_{ij} \left\| \frac{\partial V_j^{(k,l)}}{\partial \delta_j} \right\| \cdot \|D_j\| \\ & + e_{ij} \left\| \frac{\partial V_i^{(k,l+1)}}{\partial \delta_i} \right\| \cdot \|D_j\| \end{aligned} \quad (45)$$

where $\Delta v_j = v_j^{l+1} - v_j^l$, and $\underline{\sigma}(T_{ij})$ is the minimum singular value of T_{ij} . This holds if $e_{ij} = 0, \beta_{ij} = 0$. By continuity, $V_i^{(k,l)} \leq V_i^{(k+1,l)}$ holds for small values of e_{ij}, β_{ij} . Similar to Theorem 3, one can show that the equality holds if the sequence has converged, and this completes the proof.

We now prove the convergence of the outer loop. By integrating $\dot{V}_i^{(k,l)} - \dot{V}_i^{(k+1,l)}$ along the solution of

$$\begin{aligned} \dot{\delta}_i &= A\delta_i + (d_i + g_i)B_i u_i^{k+1} - \sum_{j \in N_i} e_{ij} B_j u_j^k \\ &+ (d_i + g_i)D_i v_i^l - \sum_{j \in N_i} e_{ij} D_j v_j^l \end{aligned} \quad (46)$$

and considering that all agents update their policies, one has

$$\begin{aligned}
V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(\left(\frac{\partial V_i^{(k+1,l)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^{k+1} \right. \\
&\quad - \sum_{j \in N_i} e_{ij} B_j u_j^k + (d_i + g_i)D_i v_i^l - \sum_{j \in N_i} e_{ij} D_j v_j^l) \\
&\quad - \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (A\delta_i + (d_i + g_i)B_i u_i^{k+1} - \sum_{j \in N_i} e_{ij} B_j u_j^k \\
&\quad \left. + (d_i + g_i)D_i v_i^l - \sum_{j \in N_i} e_{ij} D_j v_j^l) \right) dt
\end{aligned} \tag{47}$$

Using (30), this equation becomes

$$\begin{aligned}
V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(\frac{1}{2} u_i^{kT} R_{ii} u_i^k + \frac{1}{2} \sum_{j \in N_i} u_j^{kT} R_{ij} u_j^k \right. \\
&\quad + \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i)B_i u_i^k - \left(\frac{\partial V_i^{(k,l)}}{\partial \delta_i} \right)^T (d_i + g_i)B_i u_i^{k+1} \\
&\quad - \frac{1}{2} u_i^{(k+1)T} R_{ii} u_i^{k+1} - \frac{1}{2} \sum_{j \in N_i} u_j^{(k+1)T} R_{ij} u_j^{k+1} + \left(\frac{\partial V_i^{(k+1,l)}}{\partial \delta_i} \right)^T \times \\
&\quad \left. \sum_{j \in N_i} e_{ij} B_j u_j^{k+1} - \left(\frac{\partial V_i^{(k+1,l)}}{\partial \delta_i} \right)^T \sum_{j \in N_i} e_{ij} B_j u_j^k \right) dt
\end{aligned} \tag{48}$$

Then, substituting (32) into (48) and doing some manipulations gives

$$\begin{aligned}
V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(\frac{1}{2} (u_i^{k+1} - u_i^k)^T R_{ii} (u_i^{k+1} - u_i^k) \right. \\
&\quad + \frac{1}{2} \sum_{j \in N_i} (u_j^{k+1} - u_j^k)^T R_{ij} (u_j^{k+1} - u_j^k) - \sum_{j \in N_i} u_j^{(k+1)T} R_{ij} \times \\
&\quad \left. (u_j^{k+1} - u_j^k) + \left(\frac{\partial V_i^{(k+1,l)}}{\partial \delta_i} \right)^T \sum_{j \in N_i} e_{ij} B_j (u_j^{k+1} - u_j^k) \right) dt
\end{aligned} \tag{49}$$

A sufficient condition for $V_i^{(k,l)} \geq V_i^{(k+1,l)}$ is

$$\begin{aligned}
\frac{1}{2} \underline{\sigma}(R_{ij}) \|\Delta u_j\| &> (d_j + g_j) \rho_{ij} \left\| \frac{\partial V_j^{(k,l)}}{\partial \delta_j} \right\| \cdot \|B_j\| \\
&+ e_{ij} \left\| \frac{\partial V_i^{(k+1,l)}}{\partial \delta_i} \right\| \cdot \|B_j\|
\end{aligned} \tag{50}$$

where $\Delta u_j = u_j^{k+1} - u_j^k$, and $\underline{\sigma}(R_{ij})$ is the minimum singular value of R_{ij} . This holds if $e_{ij} = 0, \rho_{ij} = 0$. By continuity, $V_i^{(k,l)} \geq V_i^{(k+1,l)}$ holds for small values of e_{ij}, ρ_{ij} . Similar to Theorem 3, one can show that the equality holds if the sequence has converged, and this completes the proof.

Consequently, according to the above results and the principle of optimality we can say that the algorithm converges to V_i^* , the solution to the coupled HJI equation (18). ■

Remark 3. Note that Algorithm 1 converges even for the asynchronous update case, in which each agent updates her policy independently. To show this, let us divide the

neighbors of the agent i into two groups: those who update their policies (denoted by N_{i1}) and those who have fixed policies (denoted by N_{i2}). Then, using the same procedure as the proof of Theorem 3 and 4, one can get

$$\begin{aligned}
V_i^{(k,l)} - V_i^{(k+1,l)} &= \int_t^\infty \left(-\frac{1}{2} \gamma^2 (v_i^{l+1} - v_i^l)^T T_{ii} (v_i^{l+1} - v_i^l) \right. \\
&\quad - \frac{1}{2} \gamma^2 \sum_{j_2 \in N_{i2}} (v_{j_2}^{l+1} - v_{j_2}^l)^T T_{ij_2} (v_{j_2}^{l+1} - v_{j_2}^l) + \gamma^2 \sum_{j_2 \in N_{i2}} v_{j_2}^{(l+1)T} T_{ij_2} \times \\
&\quad \left. (v_{j_2}^{l+1} - v_{j_2}^l) + \sum_{j_2 \in N_{i2}} \left(\frac{\partial V_i^{(k+1,l)}}{\partial \delta_i} \right)^T e_{ij_2} D_{j_2} (v_{j_2}^{l+1} - v_{j_2}^l) \right) dt
\end{aligned}$$

The difference between this equation and (44) is that in (44) the sums are over all neighbors of agent i , while for this equation, the sums are only over those neighbors of agent i who update their policies. Therefore, the rest of the proof of the convergence of Algorithm 1 for the synchronous update case is the same as the proof of Theorem 4.

6. Online Policy Iteration for Learning the Solution to the HJI equations in Real Time

In this section, the solution to the HJI equations (18) is learned online in real time based on policy iteration Algorithm 1. An actor-critic-disturbance architecture is used to approximate the solution to these HJI equations (Abu-Khalaf, Lewis, & Huang, 2006, 2008; Modares, Lewis, & Naghibi-Sistani, 2014; Vamvoudakis, & Lewis, 2012). Specifically, each agent has three neural networks (NNs): one critic, one control actor and one disturbance. That is, for N agents, N critic neural networks (NNs) are used to approximate the value functions in the Bellman equations (30), N actor NNs are used to approximate the control policies (32) and N disturbance NNs are used to approximate the disturbance policies (31). All these $3N$ NNs are tuned simultaneously and in real time.

6.1. Critic neural network

According to the Weierstrass higher-order approximation Theorem (Abou-Khalaf & Lewis, 2005), there exist NNs such that the smooth value functions V_i are approximated using critic NNs as

$$V_i(\delta_i) = W_i^T \phi_i(\delta_i) + \varepsilon_i \tag{51}$$

where W_i are the critic NN weights, $\phi_i \in R^h$ are the critic NN activation function vectors, with h the number of neurons in the hidden layer of the critic NNs. According to the Weierstrass Theorem, the NN approximation error ε_i converges to zero uniformly as $h \rightarrow \infty$. Assuming current weight estimates $\hat{W}_i$, the outputs of the critic NNs are given by

$$\hat{V}_i = \hat{W}_i^T \phi_i \tag{52}$$

Then, the Bellman equation (30) can be approximated at each step as

$$\begin{aligned}
H_i(\delta_i, \hat{W}_i, u_i, u_{-i}) &= \frac{1}{2} \delta_i^T Q_{ii} \delta_i + \frac{1}{2} u_i^T R_{ii} u_i + \frac{1}{2} \sum_{j \in N_i} u_j^T R_{ij} u_j \\
&- \frac{1}{2} \gamma^2 v_i^T T_{ii} v_i - \frac{1}{2} \gamma^2 \sum_{j \in N_i} v_j^T T_{ij} v_j + \hat{W}_i^T \frac{\partial \phi_i}{\partial \delta_i} (A \delta_i + (d_i + g_i) B_i u_i \\
&- \sum_{j \in N_i} e_{ij} B_j u_j + (d_i + g_i) D_i v_i - \sum_{j \in N_i} e_{ij} D_j v_j) = e_{H_i}
\end{aligned} \quad (53)$$

It is desired to select $\hat{W}_i$ to minimize the square residual error

$$E_1 = \frac{1}{2} e_{H_i}^2 \quad (54)$$

Then $\hat{W}_i \rightarrow W_i$ which solves (53) in a least-squares sense and e_{H_i} becomes small. Theorem 5 gives a tuning law for the critic weights that minimizes (54).

6.2. Actor and disturbance neural networks

Based on the control update policies (16), the control policies are given as actor neural networks (Vamvoudakis, Lewis, & Hudwas 2012) as follows,

$$\hat{u}_i \equiv \hat{u}_{i+N} = -(d_i + g_i) R_{ii}^{-1} B_i^T \frac{\partial \phi_i}{\partial \delta_i} \hat{W}_{i+N} \quad (55)$$

where $\hat{W}_{i+N}$ denote the current estimated values of the ideal critic NN weights W_i . Moreover, based on the disturbance update policies (17), the disturbance neural networks are chosen as

$$\hat{v}_i \equiv \hat{v}_{i+2N} = \frac{1}{\gamma^2} (d_i + g_i) T_{ii}^{-1} D_i^T \frac{\partial \phi_i}{\partial \delta_i} \hat{W}_{i+2N} \quad (56)$$

where $\hat{W}_{i+2N}$ denote the current estimated values of the ideal critic NN weights W_i . The notation $\hat{u}_{i+N}$ and $\hat{v}_{i+2N}$ in (55) and (56) are used to keep indices straight in the proof (the first N weights are used for the critics, the second N weights are for the actors and the rest for the disturbances).

The following two theorems provide the tuning laws for the actor, critic and disturbance NNs that guarantee convergence of the synchronous online multiple agent learning PI algorithm in real time to the game saddle point solution, while guaranteeing closed-loop stability. Define the critic, actor and disturbance NN estimation errors as $\tilde{W}_i = W_i - \hat{W}_i$, $\tilde{W}_{i+N} = W_i - \hat{W}_{i+N}$ and $\tilde{W}_{i+2N} = W_i - \hat{W}_{i+2N}$.

Theorem 5. (Online zero-sum graphical games) Let the error dynamics be given by (8), and consider the zero-sum differential graphical game formulation in (11). Let the critic NN at each node be given by (52), the control input be given by actor NN (55), and the disturbance input be given by disturbance NN (56). Let tuning for the i th critic NN be provided by

$$\begin{aligned}
\dot{\hat{W}}_i &= -a_i \frac{\partial E_1}{\partial \hat{W}_i} = -a_i \frac{\sigma_{i+N}}{(1 + \sigma_{i+N}^T \sigma_{i+N})^2} [\sigma_{i+N}^T \hat{W}_i + \frac{1}{2} \delta_i^T Q_{ii} \delta_i \\
&+ \frac{1}{2} (d_i + g_i)^2 \hat{W}_{i+N}^T \bar{D}_i \hat{W}_{i+N} - \frac{1}{2\gamma^2} (d_i + g_i)^2 \hat{W}_{i+2N}^T \bar{E}_i \hat{W}_{i+2N}
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{2} \sum_{j \in N_i} (d_j + g_j)^2 \hat{W}_{j+N}^T \frac{\partial \phi_j}{\partial \delta_j} B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \frac{\partial \phi_j}{\partial \delta_j} \hat{W}_{j+N} \\
&- \frac{1}{2\gamma^2} \sum_{j \in N_i} (d_j + g_j)^2 \hat{W}_{j+2N}^T \frac{\partial \phi_j}{\partial \delta_j} D_j T_{jj}^{-1} T_{ij} T_{jj}^{-1} D_j^T \frac{\partial \phi_j}{\partial \delta_j} \hat{W}_{j+2N}
\end{aligned} \quad (57)$$

where

$$\begin{aligned}
\sigma_{i+N} &= \frac{\partial \phi_i}{\partial \delta_i} (A \delta_i + (d_i + g_i) B_i \hat{u}_{i+N} - \sum_{j \in N_i} e_{ij} B_j \hat{u}_{j+N} \\
&+ (d_i + g_i) D_i \hat{v}_{i+2N} - \sum_{j \in N_i} e_{ij} D_j \hat{v}_{j+2N})
\end{aligned}$$

Let the i th actor NN be tuned as

$$\begin{aligned}
\dot{\hat{W}}_{i+N} &= -a_{i+N} \{ (S_i \hat{W}_{i+N} - F_i \bar{\sigma}_{i+N}^T \hat{W}_i) - (d_i + g_i)^2 \bar{D}_i \times \\
&\hat{W}_{i+N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_i - \sum_{j \in N_i} (d_j + g_j)^2 \frac{\partial \phi_j}{\partial \delta_j} B_j R_{jj}^{-1} \times \\
&R_{ij} R_{jj}^{-1} B_j^T \frac{\partial \phi_j}{\partial \delta_j} \hat{W}_{i+N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_j \}
\end{aligned} \quad (58)$$

and the i th disturbance NN be tuned as

$$\begin{aligned}
\dot{\hat{W}}_{i+2N} &= -a_{i+2N} \{ (\bar{S}_i \hat{W}_{i+2N} - \bar{F}_i \bar{\sigma}_{i+N}^T \hat{W}_i) + \frac{1}{\gamma^2} (d_i + g_i)^2 \times \\
&\bar{E}_i \hat{W}_{i+2N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_i + \frac{1}{\gamma^2} \sum_{j \in N_i} (d_j + g_j)^2 \frac{\partial \phi_j}{\partial \delta_j} D_j T_{jj}^{-1} \times \\
&T_{ij} T_{jj}^{-1} D_j^T \frac{\partial \phi_j}{\partial \delta_j} \hat{W}_{i+2N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_j \}
\end{aligned} \quad (59)$$

where

$$\bar{D}_i(x) \equiv \frac{\partial \phi_i}{\partial \delta_i} B_i R_{ii}^{-1} B_i^T \frac{\partial \phi_i}{\partial \delta_i}, \bar{E}_i(x) \equiv \frac{\partial \phi_i}{\partial \delta_i} D_i T_{ii}^{-1} D_i^T \frac{\partial \phi_i}{\partial \delta_i},$$

and $m_{s_i} \equiv (\sigma_{i+N}^T \sigma_{i+N} + 1)$, $\bar{\sigma}_{i+N} = \sigma_{i+N} / (\sigma_{i+N}^T \sigma_{i+N} + 1)$ and $a_i > 0, \dots, a_{i+N} > 0, \dots, a_{i+2N} > 0$, $F_i > 0, S_i > 0, \bar{F}_i > 0, \bar{S}_i > 0$, $i \in N$ are tuning parameters. Suppose that $\bar{\sigma}_{i+N} = \sigma_{i+N} / (\sigma_{i+N}^T \sigma_{i+N} + 1)$ is persistently exciting. Then the closed-loop system states $\delta_i(t)$, the critic NN errors $\tilde{W}_i$, the actor NN errors $\tilde{W}_{i+N}$ and the disturbance NN errors $\tilde{W}_{i+2N}$ are uniformly ultimately bounded.

Proof: The proof is similar to (Vamvoudakis, Lewis, & Hudwas, 2012; Vamvoudakis, & Lewis, 2012) and is omitted.

Theorem 6. (Convergence to zero-sum Nash equilibrium) Suppose the hypotheses of Theorem 5 hold. Then:

a) $H_i(\delta_i, \hat{W}_i, \hat{u}_i, \hat{u}_{-i}, \hat{v}_i, \hat{v}_{-i}), \forall i \in N$ are uniformly ultimately bounded, that is, $\hat{W}_i$ converge to the approximate coupled HJI solution, the value of the ZS game, where

$$\hat{u}_i = -(d_i + g_i) R_{ii}^{-1} B_i^T \frac{\partial \phi_i}{\partial \delta_i} \hat{W}_i, \hat{v}_i = \frac{1}{\gamma^2} (d_i + g_i) T_{ii}^{-1} D_i^T \frac{\partial \phi_i}{\partial \delta_i} \hat{W}_i$$

b) $\hat{u}_{i+N}, \hat{v}_{i+2N}$ (see (55) and (56)) converge to the approximate Nash equilibrium solution of the zero-sum game.

Proof: The proof is similar to (Vamvoudakis, Lewis, & Hudas 2012 ; Vamvoudakis, & Lewis, 2012) and is omitted.

7. Simulation Results

In this section, the effectiveness of the online approach described in Theorem 5 is shown by a simulation example.

Fig. 1. Communication Graph.

For the digraph structure (Fig.1) shown in Vamvoudakis, Lewis, and Hudas (2012), we consider the following node dynamics

$$\dot{x}_i = Ax_i + B_i u_i + D_i v_i$$

$$\text{where } A = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, B_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, B_5 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, D_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, D_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, D_4 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, D_5 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

with target generator $\dot{x}_0 = Ax_0$.

The graphical game is implemented as in Theorem 5. Persistence of excitation was ensured by adding a small exponentially decreasing probing noise to the control inputs. The dynamics of each agent is marginally stable and so synchronization is required to a Lissajous pattern in the phase plane. Figs. 2 and 3 show the evolution of the system for agents 1, 2, 3, 4, 5 and leader 0. Fig. 4 shows the 3-D phase plane plot of all five agents and the leader node.

8. Conclusion

This paper provides a new perspective on the synchronization of multi-agent linear systems influenced by external disturbances. It brings together distributed control, optimal control theory and differential game theory to formulate and solve the multi-player zero-sum graphical games. It is shown that the solution to the given problem requires the solution to coupled HJI equations. It is also shown that the HJI solution makes the global synchronization error L_2 -bounded. A policy iteration algorithm is presented to solve coupled HJI equations and her conver-

gence to the Nash equilibrium solution of the multi-player zero-sum game is shown. An online policy iteration algorithm is given which learns the solution to these HJI equations in real time. Simulation results are provided to show the effectiveness of the proposed method.

Fig. 2. Evolution of the states for $x_{1_1}, x_{2_1}, x_{3_1}, x_{4_1}, x_{5_1}, x_{0_1}$.

Fig. 3. Evolution of the states for $x_{1_2}, x_{2_2}, x_{3_2}, x_{4_2}, x_{5_2}, x_{0_2}$.

Fig. 4. 3-D Phase Plane Plot

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