The synchronization of instantaneously coupled harmonic oscillators using sampled data with measurement noise

Jingyi Wang^a Jianwen Feng^{b,1} Chen Xu^{b,1} Michael Z. Q. Chen^c Yi Zhao^b
Jiqiang Feng^b

^aKey Laboratory of Optoelectronic Devices and Systems of Ministry of Education and Guangdong Province, and College of Optoelectronic Engineering, Shenzhen University, Shenzhen 518060, PR China

^bCollege of Mathematics and Computational Science, Shenzhen University, Shenzhen 518060, PR China

^cDepartment of Mechanical Engineering, The University of Hong Kong, Hong Kong

Abstract

In this brief, we propose an effective algorithm for synchronization of instantaneously coupled harmonic oscillators by using sampled data which may contain measurement noise. We discuss the convergence of this algorithm for both fixed and switching directed network topologies in the presence or absence of leaders. We also establish sufficient conditions under which the coupled harmonic oscillators could attain synchronized oscillatory motions in the presence of noise. And moreover, it is shown that synchronization can be attained even when the velocity information is exchanged only at discrete-time instants. Finally, we give some numerical examples to illustrate the effectiveness of the theoretical results.

Key words: Synchronization, coupled harmonic oscillators, instantaneous interactions, sampled data, measurement noise

1 Introduction

Synchronization phenomena are common in nature and hold particular promise for applications to many fields (for example, population dynamics, power systems and automatic control, see Boccaletti et al. (2002); Dörfler and Bullo (2014); Tanner et al. (2007); Wang and Chen (2002); Wang et al. (2015b)). Generally speaking, synchronization is the process in which two or more (coupled) dynamical systems seek to adjust a certain prescribed property of their motions to a common behavior in the limit as time tends to infinity (Boccaletti et al. (2002)). Many studies have therefore been conducted on these problems (see Boccaletti et al. (2002); Dörfler

and Bullo (2014); Olfati-Saber et al. (2007); Wang et al. (2015a) and the references therein) because of their great relevance to real world applications.

Coupled harmonic oscillators are interesting because both their frequencies and amplitudes can be made to converge to the same values over time, irrespective of their initial displacements (Ballard et al. (2010)). As a result, some authors have devoted to investigating different synchronization algorithms for coupled harmonic oscillators from various perspectives. For instance, Ren (2008) studied the synchronization of continuous-time coupled harmonic oscillators with local interactions over both fixed and switching network topologies, while Ballard et al. (2010) also addressed discrete-time coupled harmonic oscillators and applied it to the synchronized motions of multiple mobile robots. In addition, Zhou et al. (2012) considered the synchronization of coupled harmonic oscillators with local instantaneous interactions and obtained some sufficient conditions for fixed and switching topologies in either the presence or the absence of leaders. Zhang and Zhou (2012) also investigated the synchronization of undirected networks of coupled harmonic oscillators using sampled data measurements with controller failures. Zhou et al. (2013)

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Email addresses: jingyi.jy.wang@gmail.com (Jingyi Wang), fengjw@szu.edu.cn (Jianwen Feng), xuchen.szu@szu.edu.cn (Chen Xu), mzqchen@hku.hk (Michael Z. Q. Chen), yzhao@szu.edu.cn (Yi Zhao), fengjq@szu.edu.cn (Jiqiang Feng).

¹ Tel.: +86 755 26532312; Fax: +86 755 26538881.

further addressed sampled data synchronization of coupled harmonic oscillators with controller failure and communication delays. Furthermore, Sun et al. (2014) investigated the synchronization of instantaneously coupled harmonic oscillators with directed topology. Recently, Sun et al. (2015) focused on synchronization of coupled harmonic oscillators in the presence of random noise. In particular, the synchronization of coupled harmonic oscillators plays an important role in applications such as motion coordination, consensus seeking and collective tracking, see Casau et al. (2015); Chiorescu et al. (2004); Li and Ding (2015); Ren and Beard (2005); Ren (2008); Zhang et al. (2013); Zhou et al. (2012).

With the aid of high-speed computers and modern control systems and signal processing techniques, samples of the control input signals need only be taken at discrete-time instants. Meanwhile, many synchronization studies of dynamical networks have been performed recently using sampled data as the main means of information transfer among the network agents. For example, synchronization of coupled harmonic oscillators was considered via sampled data control by Sun et al. (2014); Zhou et al. (2013). In addition, sampled data synchronization control of dynamical networks was studied by stochastic sampling by Shen et al. (2012) and aperiodic sampling by Wen et al. (2014). No studies, however, have ever been conducted on the synchronization of instantaneously coupled harmonic oscillators using sampled data that contains measurement noise.

In this paper, we consider the dynamic behaviors of a set of identical harmonic oscillators. The main aim of this paper is to identify and analyze the synchronization conditions for instantaneously coupled harmonic oscillators using only sampled velocity data with measurement noise in either the presence or the absence of leaders. The communication topology of the proposed algorithm is directed, because the sampling cost of algorithms with directed information flow is smaller than their undirected counterparts. The convergence analysis uses tools as algebraic graph theory, matrix theory, statistics and impulsive differential equations, and the theoretical results are then applied to collective tracking of some specific multi-agent systems.

The rest of this paper is organized as follows. In Section 2, we introduce the general model for a system of instantaneously coupled harmonic oscillators with measurement noise. In Section 3, we establish some synchronization criteria for such networks when leaders are either present or absent and we illustrate the theoretical results with some numerical examples in Section 4. Finally, conclusions are drawn in Section 5.

2 Preliminaries

First, we write down the notations that will be used throughout this paper. Let $\mathbf{1}_n = [1, 1, \dots, 1]^\top \in \mathbb{R}^n$ and I_n be the n -dimensional identity matrix and we use the superscript $\top$ to denote the transpose of a matrix and $\rho(\cdot)$ denotes the spectral radius. We write $x \doteq y$ to mean that x is defined to be another name for y . Finally, for square matrices A and B , we write $A \succ B$ (respectively, $\succeq$, $\prec$ and $\preceq$) to mean that $A - B$ is positive definite (respectively, positive semi-definite, negative definite and negative semi-definite).

To analyze the convergence conditions for a set of coupled harmonic oscillators over any arbitrary network topology, the oscillator interactions shall be modeled by a directed graph. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph for some set of nodes $\mathcal{V} = \{1, 2, \dots, n\}$ and set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. And denote $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ as the adjacency matrix associated with $\mathcal{G}$ such that a_{ij} is a positive weight for all $(i, j) \in \mathcal{E}$ (so that node i receives information from node j while j being called the parent of i) and $a_{ij} = 0$ for all $(i, j) \notin \mathcal{E}$. Define the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ associated with $\mathcal{G}$ by $l_{ii} = \sum_{j=1, j \neq i}^n l_{ij}$, where $l_{ij} = -a_{ij} (\leq 0)$ for $i \neq j$. A sequence of edges $(i_1, i_2), (i_2, i_3), \dots$, for $i_j \in \mathcal{V}$, is called a directed path of $\mathcal{G}$. And a directed graph is strongly connected if there is a directed path from every node to every other node. Moreover, a directed graph is balanced if $\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$ for every node i .

The measured values used in the sampled data control methods (Zhang and Shi (2012)), however, are usually contaminated with measurement noise arising from the measuring errors of the physical devices and the various states of the observers (for instance, the observers' relative positions to the instruments and so on). In Fig. 1, let the points $j(x_j, y_j)$, $i(x_i, y_i)$ and $\tilde{i}(x_{\tilde{i}}, y_{\tilde{i}})$ be the state of the observer, the state of the (expected) observed node and the actual measured state of the observed node, respectively. Let $\vec{j\tilde{i}}$, $\vec{j\hat{i}}$ and $\vec{i\tilde{i}}$ be the real difference, the measured difference and the measurement noise, respectively, and $\omega = [\omega_x, \omega_y]^\top \in \mathbb{R}^2$ be a random noise vector. Clearly, $\vec{j\tilde{i}} = \vec{j\hat{i}} + \vec{i\tilde{i}}$. More precisely, we have

$$\begin{bmatrix} x_{\tilde{i}} - x_j \\ y_{\tilde{i}} - y_j \end{bmatrix} = \begin{bmatrix} x_i - x_j \\ y_i - y_j \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} \circ \begin{bmatrix} x_i - x_j \\ y_i - y_j \end{bmatrix},$$

where $\circ$ is the Hadamard product.

Motivated by the aforementioned comments and taking sampled data noise into account, we propose a highly effective algorithm for synchronizing n instantaneously coupled harmonic oscillators that are described by the

Fig. 1. A 2D sketch of sampled data containing measurement noise.

impulsive differential equations

$$\begin{cases} \dot{r}_i(t) = v_i(t), & \dot{v}_i(t) = -\alpha r_i(t), & t \neq t_k, \\ \Delta r_i(t_k) = 0, & & t = t_k, \\ \Delta v_i(t_k) = \sum_{j=1}^n a_{ij} [\mu(v_j(t_k) - v_i(t_k)) \\ + \omega_k(v_j(t_k) - v_i(t_k))], & & t = t_k, \end{cases} \quad (1)$$

$i = 1, 2, \dots, n$, for some positive constant α , where $r(t) \in \mathbb{R}$ is the position of the oscillator and $v(t) \in \mathbb{R}$ is its velocity, the coupling strength $\mu > 0$, the coefficients $a_{ij} \geq 0$ if and only if oscillator i can attain the velocity of oscillator j , the time sequence $\{t_k\}_{k=1}^{+\infty}$ satisfying $0 = t_0 < t_1 < \dots < t_k < \dots$, $\lim_{k \rightarrow \infty} t_k = +\infty$, $\Delta r_i(t_k) = r_i(t_k^+) - r_i(t_k)$, $\Delta v_i(t_k) = v_i(t_k^+) - v_i(t_k)$ with $r_i(t_k^+) = \lim_{t \rightarrow t_k^+} r_i(t)$, $v_i(t_k^+) = \lim_{t \rightarrow t_k^+} v_i(t)$, $\lim_{t \rightarrow t_k^-} r_i(t) = r_i(t_k)$ and $\lim_{t \rightarrow t_k^-} v_i(t) = v_i(t_k)$ (so that $r_i(t)$ and $v_i(t)$ are left continuous), the sequence of random variables $\{\omega_k\}$ is independent and identically distributed with zero-mean and σ^2 variance and the part $\omega_k(v_i(t_k) - v_j(t_k))$ is the measurement noise and depends on the states of oscillators i and j for $k \in \mathbb{N}$.

3 Synchronization analysis

3.1 Synchronization without a leader

Theorem 1 *If the directed graph $\mathcal{G}$ is strongly connected and*

- (1) $0 < \lambda_n < \frac{(1 - 1/\alpha) \sin^2(\sqrt{\alpha} T_k) \zeta_n + 1 - \zeta_n}{(\alpha - 1) \sin^2(\sqrt{\alpha} T_k) + 1 - \zeta_n}$;
- (2) $\alpha > 1 + \frac{\zeta_n - 1}{1 - \zeta_n + \zeta_n \sin^2(\sqrt{\alpha} T_k)} > 0$;
- (3) $T_k \neq \frac{j\pi}{\sqrt{\alpha}}$, for all $k \in \mathbb{N}$ and $j \in \mathbb{N}$,

where $\zeta_n > 0$ and $\lambda_n > 0$ are the greatest eigenvalues of $(I_n - \Xi)^\top (I_n - \Xi)$ and $(I_n - \Xi - \mu L)^\top (I_n - \Xi - \mu L) + \sigma^2 L^\top L$, respectively, then the solutions $[r_i(t), v_i(t)]^\top$ of system (1) converge in mean square to the synchronized

state

$$\begin{bmatrix} \gamma(t) \\ \nu(t) \end{bmatrix} = \begin{bmatrix} \xi^\top r(0) \cos(\sqrt{\alpha} t) + \frac{1}{\sqrt{\alpha}} \xi^\top v(0) \sin(\sqrt{\alpha} t) \\ -\sqrt{\alpha} \xi^\top r(0) \sin(\sqrt{\alpha} t) + \xi^\top v(0) \cos(\sqrt{\alpha} t) \end{bmatrix}, \quad (2)$$

where $\xi = [\xi_1, \xi_2, \dots, \xi_n]^\top$ ($\sum_{i=1}^n \xi_i = 1$) is a left eigenvector of L corresponding to the eigenvalue 0, $\Xi = \mathbf{1}_n \xi^\top$ and $r(0)$ and $v(0)$ are the initial values of system (1).

PROOF. Rewrite equation (1) into the following form

$$\begin{cases} \begin{bmatrix} \dot{r}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} O_n & I_n \\ -\alpha I_n & O_n \end{bmatrix} \begin{bmatrix} r(t) \\ v(t) \end{bmatrix}, & t \neq t_k \\ \begin{bmatrix} r(t_k^+) \\ v(t_k^+) \end{bmatrix} = \begin{bmatrix} I_n & O_n \\ O_n & I_n - \mu L - \omega_k L \end{bmatrix} \begin{bmatrix} r(t_k) \\ v(t_k) \end{bmatrix}, & t = t_k \end{cases} \quad (3)$$

and let $[r^\top(t), v^\top(t)]^\top$ be the solution of (3) with initial value $[r^\top(0), v^\top(0)]^\top$. Then write

$$\begin{bmatrix} \dot{\gamma}(t) \\ \dot{\nu}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} \gamma(t) \\ \nu(t) \end{bmatrix} \quad (4)$$

and, noting that $[\gamma(t), \nu(t)]^\top$ in (2) is the solution of (4) with initial value $[\xi^\top r(0), \xi^\top v(0)]^\top$, introduce the synchronization error

$$\begin{bmatrix} e(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ v(t) \end{bmatrix} - \begin{bmatrix} \gamma(t) \mathbf{1}_n \\ \nu(t) \mathbf{1}_n \end{bmatrix}. \quad (5)$$

It is clear, from equations (3) and (4), that $[e^\top(t), s^\top(t)]^\top$ is a solution of equation (3) corresponding to the initial value

$$\begin{bmatrix} e(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} I_n - \Xi & O_n \\ O_n & I_n - \Xi \end{bmatrix} \begin{bmatrix} r(0) \\ v(0) \end{bmatrix}$$

and so it follows from the theory of impulsive differential equations (Yang (2001)) that

$$\begin{aligned} \begin{bmatrix} e(t) \\ s(t) \end{bmatrix} &= \exp \left(\begin{bmatrix} O_n & I_n \\ -\alpha I_n & O_n \end{bmatrix} (t - t_{k-1}) \right) \\ &\times \prod_{j=1}^{k-1} \left\{ \begin{bmatrix} I_n & O_n \\ O_n & I_n - (\mu + \omega_{k-j}) L \end{bmatrix} \exp \left(\begin{bmatrix} O_n & I_n \\ -\alpha I_n & O_n \end{bmatrix} T_{k-j} \right) \right\} \\ &\times \begin{bmatrix} I_n - \Xi & O_n \\ O_n & I_n - \Xi \end{bmatrix} \begin{bmatrix} r(0) \\ v(0) \end{bmatrix}, \quad t \in (t_{k-1}, t_k], \quad k \geq 2, \end{aligned}$$

where

$$\exp \left(\begin{bmatrix} O_n & I_n \\ -\alpha I_n & O_n \end{bmatrix} s \right) = \begin{bmatrix} \cos(\sqrt{\alpha}s)I_n & \frac{1}{\sqrt{\alpha}} \sin(\sqrt{\alpha}s)I_n \\ -\sqrt{\alpha} \sin(\sqrt{\alpha}s)I_n & \cos(\sqrt{\alpha}s)I_n \end{bmatrix}. \quad \mathbb{E} \left\{ \begin{bmatrix} e(t_1^+) \\ s(t_1^+) \end{bmatrix} \right\} = \mathbb{E}[B_1] \exp(AT_1) \begin{bmatrix} r(0) \\ v(0) \end{bmatrix} \quad (7)$$

with the expected value

and the covariance matrix

$$\text{cov} \left\{ \begin{bmatrix} e(t_1^+) \\ s(t_1^+) \end{bmatrix} \right\} = \mathbb{E} \left\{ \omega_1 \tilde{L} \exp(AT_1) \begin{bmatrix} r(0) \\ v(0) \end{bmatrix} \times \begin{bmatrix} r(0) \\ v(0) \end{bmatrix}^\top \exp(A^\top T_1) \omega_1 \tilde{L}^\top \right\},$$

Now we know, by definition of Ξ and L , that $\Xi L = L \Xi = O_n$, $\Xi^2 = \Xi(I_n - \Xi)^2 = I_n - \Xi$, $(I_n - L)(I_n - \Xi) = I_n - L - \Xi$ and $(I_n - \Xi)(I_n - L - \Xi) = I_n - L - \Xi$ and that $[e^\top(t), s^\top(t)]^\top$ is just a solution of the impulsive differential equations (with noise)

$$\begin{cases} \begin{bmatrix} \dot{e}(t) \\ \dot{s}(t) \end{bmatrix} = \begin{bmatrix} O_n & I_n \\ -\alpha I_n & O_n \end{bmatrix} \begin{bmatrix} e(t) \\ s(t) \end{bmatrix}, & t \neq t_k, \\ \begin{bmatrix} e(t_k^+) \\ s(t_k^+) \end{bmatrix} = \begin{bmatrix} I_n - \Xi & O_n \\ O_n & I_n - \Xi - \mu L - \omega_k L \end{bmatrix} \begin{bmatrix} e(t) \\ s(t) \end{bmatrix}, & t = t_k, \end{cases} \quad \text{and by mathematical calculations (see Section 6.2 of Petersen and Pedersen (2012)), one can get}$$

with initial condition $[e(0)^\top, s(0)^\top] = [r(0)^\top, v(0)^\top]$ (see Zhou et al. (2012)). Therefore, by letting

$$A = \begin{bmatrix} O_n & I_n \\ -\alpha I_n & O_n \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} O_n & O_n \\ O_n & L \end{bmatrix},$$

$$B = \begin{bmatrix} I_n - \Xi & O_n \\ O_n & I_n - \Xi - \mu L \end{bmatrix}$$

and $B_k = B - \omega_k \tilde{L}$, one can get $\mathbb{E}[B_k] = B$ and $\mathbb{E}[B_k^\top B_k] = \text{diag}\{B_{11}, B_{22}\}$, where $B_{11} = (I_n - \Xi)^\top (I_n - \Xi)$ and $B_{22} = (I_n - \Xi - \mu L)^\top (I_n - \Xi - \mu L) + \sigma^2 L^\top L$. So that, for all $t \in [t_{k-1}, t_k]$, we have

$$\begin{cases} e_i(t) = \cos(\sqrt{\alpha}t) e_i(t_{k-1}^+) + \frac{1}{\sqrt{\alpha}} \sin(\sqrt{\alpha}t) s_i(t_{k-1}^+), \\ s_i(t) = -\sqrt{\alpha} \sin(\sqrt{\alpha}t) e_i(t_{k-1}^+) + \cos(\sqrt{\alpha}t) s_i(t_{k-1}^+). \end{cases}$$

Furthermore, it is clear that

$$e_i(t)^\top e_i(t) + \frac{1}{\alpha} s_i(t)^\top s_i(t) = \Delta_{i,k}, \quad \text{for } t \in [t_{k-1}, t_k], \quad (6)$$

where $\Delta_{i,k} = e_i(t_{k-1}^+)^\top e_i(t_{k-1}^+) + \frac{1}{\alpha} s_i(t_{k-1}^+)^\top s_i(t_{k-1}^+)$. It now suffices to show that $\sum_{i=1}^n \Delta_{i,k} \rightarrow 0$ as $k \rightarrow +\infty$ achieves synchronization.

When $t = t_1^+$, we have

$$\begin{bmatrix} e(t_1^+) \\ s(t_1^+) \end{bmatrix} = B_1 \exp(AT_1) \begin{bmatrix} r(0) \\ v(0) \end{bmatrix}$$

and by mathematical calculations (see Section 6.2 of Petersen and Pedersen (2012)), one can get

$$\begin{aligned} & \mathbb{E} \left\{ \begin{bmatrix} e(t_1^+) \\ s(t_1^+) \end{bmatrix}^\top \begin{bmatrix} e(t_1^+) \\ s(t_1^+) \end{bmatrix} \right\} \\ & \leq \rho \left(\mathbb{E} \left\{ e^{A^\top T_1} B_1^\top B_1 e^{AT_1} \right\} \right) \begin{bmatrix} r(0) \\ v(0) \end{bmatrix}^\top \begin{bmatrix} r(0) \\ v(0) \end{bmatrix}. \end{aligned}$$

When $t = t_2^+$, in the similar way, we obtain

$$\begin{aligned} & \mathbb{E} \left\{ \begin{bmatrix} e(t_2^+) \\ s(t_2^+) \end{bmatrix}^\top \begin{bmatrix} e(t_2^+) \\ s(t_2^+) \end{bmatrix} \right\} \\ & \leq \rho \left(\mathbb{E} \left\{ e^{A^\top T_2} B_2^\top B_2 e^{AT_2} \right\} \right) \mathbb{E} \left\{ \begin{bmatrix} e(t_1^+) \\ s(t_1^+) \end{bmatrix}^\top \begin{bmatrix} e(t_1^+) \\ s(t_1^+) \end{bmatrix} \right\}. \end{aligned}$$

It therefore follows by induction that

$$\begin{aligned} & \mathbb{E} \left\{ \begin{bmatrix} e(t_k^+) \\ s(t_k^+) \end{bmatrix}^\top \begin{bmatrix} e(t_k^+) \\ s(t_k^+) \end{bmatrix} \right\} \\ & \leq \rho \left(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \right) \mathbb{E} \left\{ \begin{bmatrix} e(t_{k-1}^+) \\ s(t_{k-1}^+) \end{bmatrix}^\top \begin{bmatrix} e(t_{k-1}^+) \\ s(t_{k-1}^+) \end{bmatrix} \right\} \end{aligned} \quad (8)$$

and we must now show that $\rho \left(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \right) < 1$.

There exist unitary matrices U and V such that $U^\top (I_n - \Xi)^\top (I_n - \Xi) U = \text{diag}\{0, \zeta_2, \dots, \zeta_n\} = H$ and $V^\top ((I_n - \Xi - \mu L)^\top (I_n - \Xi - \mu L) + \sigma^2 L^\top L) V = \text{diag}\{0, \lambda_2, \dots, \lambda_n\} = \Lambda$, $\zeta_i \geq \zeta_{i-1} > 0$, $\lambda_i \geq \lambda_{i-1} > 0$ for $i = 3, \dots, n$. By letting $\hat{\Lambda} = \text{diag}\{0, \lambda_n, \dots, \lambda_n\} \in \mathbb{R}^{n \times n}$ and $\hat{H} = \text{diag}\{0, \zeta_n, \dots, \zeta_n\}$, we have

$$\text{diag} \{U^\top, V^\top\} \mathbb{E} \{B_k^\top B_k\} \text{diag} \{U, V\}$$

$$= \begin{bmatrix} H & O_n \\ O_n & \Lambda \end{bmatrix} \preceq \begin{bmatrix} \tilde{H} & O_n \\ O_n & \tilde{\Lambda} \end{bmatrix} \doteq \tilde{B}.$$

Now, it is clear that $\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \preceq e^{A^\top T_k} \tilde{B} e^{AT_k}$ and $\rho(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\}) \leq \rho(e^{A^\top T_k} \tilde{B} e^{AT_k})$ and so the characteristic equation of $e^{A^\top T_k} \tilde{B} e^{AT_k}$ is

$$\begin{aligned} & \det \left(e^{A^\top T_k} \tilde{B} e^{AT_k} - x I_{2n} \right) \\ &= x^2 \prod_{i=2}^n \left\{ x^2 - \left(\frac{1}{\alpha} \zeta_n \sin^2(\sqrt{\alpha} T_k) + \zeta_n \cos^2(\sqrt{\alpha} T_k) \right) \right. \\ & \quad \left. + \lambda_n \cos^2(\sqrt{\alpha} T_k) + \alpha \lambda_n \sin^2(\sqrt{\alpha} T_k) \right\} x + \lambda_n \zeta_n \end{aligned}$$

with roots (i.e. eigenvalues) $\lambda_{(1)\pm} \left(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \right) = 0$ and $\lambda_{(i)\pm} \left(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \right) = \frac{1}{2} (b \pm \sqrt{b^2 - 4\lambda_n \zeta_n})$, $i = 2, 3, \dots, n$, where $b = \frac{1}{\alpha} \zeta_n \sin^2(\sqrt{\alpha} T_k) + \zeta_n \cos^2(\sqrt{\alpha} T_k) + \lambda_n \cos^2(\sqrt{\alpha} T_k) + \alpha \lambda_n \sin^2(\sqrt{\alpha} T_k)$. Since

$$\left| \lambda_{(i)\pm} \left(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \right) \right| = \frac{1}{2} \left(b \pm \sqrt{b^2 - 4\lambda_n \zeta_n} \right),$$

when $b^2 \geq 4\lambda_n \zeta_n$ and

$$\left| \lambda_{(i)\pm} \left(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \right) \right| = \sqrt{\lambda_n \zeta_n},$$

when $b^2 < 4\lambda_n \zeta_n$, we have $\left| \lambda_{(i)\pm} \left(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\} \right) \right| < 1$ when $b < 1 + \lambda_n \zeta_n$ and $\lambda_n \zeta_n < 1$.

Also, it is clear that $((\frac{1}{\alpha} - 1) \sin^2(\sqrt{\alpha} T_k) \zeta_n + \zeta_n - 1) \zeta_n < ((\alpha - 1) \sin^2(\sqrt{\alpha} T_k) - \zeta_n + 1)$ and $1 + \frac{\zeta_n - 1}{\sin^2(\sqrt{\alpha} T_k)} \leq 1 + \frac{\zeta_n - 1}{1 - \zeta_n + \zeta_n \sin^2(\sqrt{\alpha} T_k)}$ for any $\alpha \in \mathbb{R}$, $\zeta_n \geq 1$ and $\sqrt{\alpha} T_k \neq j\pi$, $j \in \mathbb{N}$ with the equality holding only when $\zeta_n = 1$. Therefore, since $b < 1 + \lambda_n \zeta_n$ and $\lambda_n \zeta_n < 1$ only when conditions (1), (2) and (3) hold, we see that $\rho(\mathbb{E} \left\{ e^{A^\top T_k} B_k^\top B_k e^{AT_k} \right\}) \leq \gamma < 1$ and we have

$$\lim_{k \rightarrow \infty} \mathbb{E} \left\{ \begin{bmatrix} e(t_k^+) \\ s(t_k^+) \end{bmatrix}^\top \begin{bmatrix} e(t_k^+) \\ s(t_k^+) \end{bmatrix} \right\} \leq \lim_{k \rightarrow \infty} \gamma^k \begin{bmatrix} r(0) \\ v(0) \end{bmatrix}^\top \begin{bmatrix} r(0) \\ v(0) \end{bmatrix} = 0. \quad (9)$$

It now follows from (6) that

$$\lim_{t \rightarrow \infty} \mathbb{E} \left\{ \begin{bmatrix} e(t) \\ s(t) \end{bmatrix}^\top \begin{bmatrix} e(t) \\ s(t) \end{bmatrix} \right\} = 0$$

and so all the states $(r_i(t), v_i(t))^\top$, $i = 1, 2, \dots, n$ of (1) will asymptotically converge in mean square to the synchronized state $(\gamma(t), \nu(t))^\top$. This completes the proof of Theorem 1.

Remark 2 In this study, the communication topology $\mathcal{G}$ does not need to be undirected and its Laplace matrix L is asymmetric. One cannot directly analyze and compare converges by using its eigenvalues, since they may be complex scale. We deal with the problem by using the singular value decomposition, which is less often used to study synchronization in the literature, but is nevertheless extremely useful. Sufficient conditions for synchronization of coupled harmonic oscillators are given in Theorem 1. Because measurement noise directly affects synchronization, it is crucial to find the relationship between the coupling and noise strengths. We can readily compute the values of μ and σ for any given graph $\mathcal{G}$ with the associated Laplacian matrix L . We will also give an example of the relationship between μ and σ^2 in Fig. 3.

Remark 3 Compared with the existing work concerning synchronization schemes for coupled harmonic oscillators (such as, sampled data control without noise by Sun et al. (2014); Zhou et al. (2012), and continuous-time coupling without noise by Ren (2008) or with noise in Sun et al. (2015)), a great advantage of the proposed algorithm (1) is that it can deal with the problem based only on the neighbors' relative velocity information in the presence of measurement noise at discrete-time instants, which does not need exact and continuous-time position and velocity information.

We also consider the convergence of coupled harmonic oscillators over switching directed communication topologies without leaders. Let $S_{\mathcal{G}} = \{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ be the set of all possible interconnection graphs at the coupling instant $t = t_k$ for $k \in \mathbb{N}$ and let $G_{s(k)} \in S_{\mathcal{G}}$ be the communication topology at $t = t_k$ with the corresponding Laplacian $L_{s(k)}$ and adjacent matrix $A_{s(k)} = [a_{ij}^{s(k)}] \in \mathbb{R}^{n \times n}$, where the map $s(k) : \mathbb{N} \rightarrow \mathcal{I} = \{1, 2, \dots, m\}$ is a stochastic switching signal that determines the network topology $\mathcal{G}_{s(k)}$ at $t = t_k$ for $k \in \mathbb{N}$. In the case we consider the following algorithm

$$\begin{cases} \dot{r}_i(t) = v_i(t), & \dot{v}_i(t) = -\alpha r_i(t), & t \neq t_k, \\ \Delta r_i(t_k) = 0, & & t = t_k, \\ \Delta v_i(t_k) = \sum_{j=1}^n a_{ij}^{s(k)} (\mu + \omega_k) (v_j(t_k) - v_i(t_k)), & & t = t_k. \end{cases} \quad (9)$$

where $i = 1, 2, \dots, n$. Then, we have the following theorem for the algorithm (9).

Theorem 4 If the directed graphs $\mathcal{G}_1, \dots, \mathcal{G}_m$ are balanced and strongly connected and

$$0 < \lambda_{n, \mathcal{I}} < \frac{1}{\alpha}, \quad \alpha > 1 \quad \text{and} \quad T_k \neq \frac{j\pi}{\sqrt{\alpha}}$$

for all $k \in \mathbb{N}$ and $j \in \mathbb{N}$, where $\lambda_{n, \mathcal{I}} > 0$ are the greatest

eigenvalues of $(I_n - \Xi - \mu L_s)^\top (I_n - \Xi - \mu L_s) + \sigma^2 L_s^\top L_s$ for all $s = 1, 2, \dots, m$, then the solutions $[r_i(t), v_i(t)]^\top$ of system (9) will converge in mean square to the synchronized state $[\gamma(t), \nu(t)]^\top$ in equation (2), where $\xi = [1/n, 1/n, \dots, 1/n]^\top$ and $r(0)$ and $v(0)$ are the initial values of system (9).

Remark 5 If the directed graphs $\mathcal{G}_s$ with the corresponding Laplacian L_s are balanced and strongly connected, then $\xi = [1/n, 1/n, \dots, 1/n]^\top$ is a left eigenvector of L_s corresponding to the eigenvalue 0 for all $s \in \mathcal{I}$. And $(I_n - \Xi)^\top (I_n - \Xi)$ is diagonalizable with a diagonal form $\text{diag}\{0, 1, 1, \dots, 1\}$. Based on above, Theorem 4 follows directly from Theorem 1 and the proof is thus omitted for brevity.

3.2 Synchronization with a leader

In this subsection, we investigate the situation when an active leader exists with position $\hat{\gamma}(t)$ and velocity $\hat{\nu}(t)$ given by

$$\dot{\hat{\gamma}}(t) = \hat{\nu}(t), \quad \dot{\hat{\nu}}(t) = -\alpha \hat{\gamma}(t). \quad (10)$$

In this case, we propose the algorithm

$$\begin{cases} \dot{r}_i(t) = v_i(t), & \dot{v}_i(t) = -\alpha r_i(t), & t \neq t_k, \\ \Delta r_i(t_k) = 0, & & t = t_k, \\ \Delta v_i(t_k) = (\mu + \omega_k) \left(\sum_{j=1}^n a_{ij}(v_j(t_k) - v_i(t_k) - d_i(v_i(t_k) - \hat{\nu}(t_k))) \right), & & t = t_k, \end{cases} \quad (11)$$

for $i = 1, 2, \dots, n$, where $d_i > 0$ if $\hat{\nu}(t_k)$ is available to oscillator i at $t = t_k$ ($k \in \mathbb{N}$) and $d_i = 0$ otherwise.

Theorem 6 If the directed graph $\mathcal{G}$ is strongly connected and

$$0 < \hat{\lambda}_n < \frac{1}{\alpha}, \quad \alpha > 1 \quad \text{and} \quad T_k \neq \frac{j\pi}{\sqrt{\alpha}} \quad (12)$$

for all $k \in \mathbb{N}$ and $j \in \mathbb{N}$, where $\hat{\lambda}_n > 0$ is the greatest eigenvalue of $(I_n - \mu(L - D))^\top (I_n - \mu(L - D)) + \sigma^2(L - D)^\top (L - D)$, then the solutions $[r_i(t), v_i(t)]^\top$ of system (11) will converge in mean square to the synchronized state $[\hat{\gamma}(t), \hat{\nu}(t)]^\top$ for any initial values.

PROOF. Consider the synchronization error

$$\begin{bmatrix} e(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ v(t) \end{bmatrix} - \begin{bmatrix} \hat{\gamma}(t) \mathbf{1}_n \\ \hat{\nu}(t) \mathbf{1}_n \end{bmatrix},$$

one can get

$$\begin{bmatrix} e(t) \\ s(t) \end{bmatrix} = e^{A(t-t_{k-1})} \otimes I_n \left\{ \prod_{j=1}^{k-1} \hat{B}_k e^{AT_{k-j}} \right\} \begin{bmatrix} e(0) \\ s(0) \end{bmatrix},$$

for $t \in (t_{k-1}, t_k]$, $k \geq 2$, where

$$\hat{B}_k = \begin{bmatrix} I_n & O_n \\ O_n & I_n - (\mu + \omega_k)(L - D) \end{bmatrix},$$

$[e(0)^\top, s(0)^\top] = [(r(0) - \hat{\gamma}(0)\mathbf{1}_n)^\top, (v(0) - \hat{\nu}(0)\mathbf{1}_n)^\top]$, $D = \text{diag}\{d_1, d_2, \dots, d_n\}$ and one can get

$$\begin{aligned} & \mathbb{E} \left\{ \begin{bmatrix} e(t_k^+) \\ s(t_k^+) \end{bmatrix}^\top \begin{bmatrix} e(t_k^+) \\ s(t_k^+) \end{bmatrix} \right\} \\ & \leq \rho \left(\mathbb{E} \left\{ e^{A^\top T_k} \hat{B}_k^\top \hat{B}_k e^{AT_k} \right\} \right) \mathbb{E} \left\{ \begin{bmatrix} e(t_{k-1}^+) \\ s(t_{k-1}^+) \end{bmatrix}^\top \begin{bmatrix} e(t_{k-1}^+) \\ s(t_{k-1}^+) \end{bmatrix} \right\}. \end{aligned}$$

Similarly, we have

$$I_2 \otimes \hat{V}^\top \mathbb{E} \left\{ \hat{B}_k^\top \hat{B}_k \right\} I_2 \otimes \hat{V} = \begin{bmatrix} I_n & O_n \\ O_n & \hat{\Lambda} \end{bmatrix} \preceq \begin{bmatrix} I_n & O_n \\ O_n & \hat{\lambda}_n I_n \end{bmatrix},$$

where $\hat{\Lambda}$ is a diagonal matrix of the eigenvalues of $(I_n - \mu(L - D))^\top (I_n - \mu(L - D)) + \sigma^2(L - D)^\top (L - D)$ with $\hat{\lambda}_n$ being the greatest one. If condition (12) of Theorem 6 holds, then we have $\rho \left(\mathbb{E} \left\{ e^{A^\top T_k} \hat{B}_k^\top \hat{B}_k e^{AT_k} \right\} \right) < 1$ and the rest of the proof is similar to that of Theorem 1. The proof is complete.

Consider the situation where there exists at least one oscillator that is connected to the active leader at $t = t_k$ ($k \in \mathbb{N}$) and that different oscillators may be connected to the leader at different instants. In this case, we propose the algorithm

$$\begin{cases} \dot{r}_i(t) = v_i(t), & \dot{v}_i(t) = -\alpha r_i(t), & t \neq t_k, \\ \Delta r_i(t_k) = 0, & & t = t_k, \\ \Delta v_i(t_k) = (\mu + \omega_k) \left(\sum_{j=1}^n a_{ij}^{s(k)} (v_j(t_k) - v_i(t_k) - d_i^{s(k)} (v_i(t_k) - \hat{\nu}(t_k))) \right), & & t = t_k, \end{cases} \quad (13)$$

for $i = 1, 2, \dots, n$.

Theorem 7 If the directed graphs $\mathcal{G}_1, \dots, \mathcal{G}_m$ are balanced and strongly connected and

$$0 < \hat{\lambda}_{n, \mathcal{I}} < \frac{1}{\alpha}, \quad \alpha > 1 \quad \text{and} \quad T_k \neq \frac{j\pi}{\sqrt{\alpha}}$$

for all $k \in \mathbb{N}$ and $j \in \mathbb{N}$, where $\hat{\lambda}_{n,\mathcal{I}} > 0$ is the greatest eigenvalue of $(I_n - \mu(L_s - D_s))^\top (I_n - \mu(L_s - D_s)) + \sigma^2(L_s - D_s)^\top (L_s - D_s)$ for all $s \in \mathcal{I}$, then the solutions $[r_i(t), v_i(t)]^\top$ of system (13) will converge in mean square to the synchronized state $[\hat{\gamma}(t), \hat{\nu}(t)]^\top$ for any initial values.

Remark 8 Theorem 7 follows directly from Theorems 4 and 6, and the proof is thus omitted for brevity.

4 Numerical Simulations

In this section, we present some numerical simulations that validate the effectiveness of the theorems in the previous section.

Fig. 2. Directed graph $\mathcal{G}$ and directed and balanced graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. We assume that $a_{ij} = 1$ for all $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise.

4.1 Synchronization of coupled harmonic oscillators

Here we consider a directed network of 10 agents moving in one-dimensional Euclidean space with the topology $\mathcal{G}$ as shown in Fig. 2(a) under algorithm (1). Clearly, graph $\mathcal{G}$ is strongly connected and we choose $\alpha = 1.2$, $T_k = 1.3$ and $\mu = 0.2$ (the set of all the accepted combinations of μ and σ^2 will be given in Fig. 3). Direct computation yields $\rho(e^{A^\top T_k} B_k^\top B_k e^{AT_k}) = 0.9711$ if $\sigma^2 = 0.003$ so that the conditions of Theorem 1 are satisfied. The evolution of the oscillator states are shown in Fig. 4(a) and Fig. 4(b) for the initial values $r_i(0) = 8i - 36$, $v_i(0) = 8i - 32$, $i = 1, 2, \dots, 10$ and the synchronization can be achieved. While one can observe that the synchronization can not be reached with the same parameters except for $\sigma^2 = 0.05$ from Fig. 4(c) and Fig. 4(d). One can see that the measurement noise affects synchronization directly, which cannot be reached when the noise standard deviation is relatively large from this example.

Next, the interaction topology is randomly switched between $\mathcal{G}_1$ and $\mathcal{G}_2$ (shown as Figs. 2(b) and (2(c))) at time t_k and we choose $\alpha = 1.2$, $T_k = 1.3$, $\sigma^2 = 0.003$ and $\mu = 0.2$. We have $\rho(e^{A^\top T_k} B_1^\top B_1 e^{AT_k}) = 0.8393$ and $\rho(e^{A^\top T_k} B_2^\top B_2 e^{AT_k}) = 0.8385$. Fig. 5 shows the evolution of the oscillator states under algorithm (9) for the initial values $r_i(0) = 8i - 36$ and $v_i(0) = 8i - 32$, $i = 1, 2, \dots, 10$.

Fig. 3. The relation between coupling and noise strength for $\mathcal{G}$ when $\alpha = 1.2$ and $T_k = 1.3$. The yellow region contains all the accepted combinations such that $\rho(e^{A^\top T_k} B_k^\top B_k e^{AT_k}) < 1$.

Fig. 4. The time evolution of r_i and v_i with a fixed interconnection topology for $\mathcal{G}$ and same initial values when $\alpha = 1.2$, $T_k = 1.3$, $\mu = 0.2$, and $\sigma^2 = 0.003$ (a,b) or $\sigma^2 = 0.05$ (c,d).

Fig. 5. The time evolution of r_i and v_i for switching interconnection topologies $\mathcal{G}_1$ and $\mathcal{G}_2$ when $\alpha = 1.2$, $T_k = 1.3$, $\sigma^2 = 0.003$ and $\mu = 0.2$.

4.2 Application to collective tracking

In the subsection, we apply algorithm (11) to collective tracking a moving object, as implemented by Casau et al. (2015); Li and Ding (2015); Zhang et al. (2013, 2015).

Consider a group of agents with the communication topology as shown in Fig. 2(a). The dynam-

ics of the object is described by (10) with initial value $\gamma(0) = 10$ and $\nu(0) = -10$. Only a subset of agents (7, 8, 9, 10) can access the velocity information of the object. Choose $\alpha = 1.2$, $\sigma^2 = 0.003$, $\mu = 0.2$, $T_k = 1.3$, $d_1 = \dots = d_6 = 0$ and $d_7 = \dots = d_{10} = 4.6$. The conditions of Theorem 6 are satisfied and $\rho(\mathbb{E}(e^{A^\top T_k} \hat{B}_k^\top \hat{B}_k e^{AT_k})) = 0.9982$. Fig. 6 demonstrates the tracking errors evolving with time. It is obvious that the collective tracking can be achieved, which shows the effectiveness of algorithm (11).

Fig. 6. The time evolution of tracking errors $r_i - \hat{\gamma}$ and $v_i - \hat{\nu}$ with a fixed communication topology $\mathcal{G}$ via tracking a moving object (10) with initial value $\hat{\gamma}(0) = 10$ and $\hat{\nu}(0) = -10$, where $\alpha = 1.2$, $T_k = 1.3$, $\sigma^2 = 0.003$ and $\mu = 0.2$.

5 Conclusion

In this paper, an effective algorithm has been proposed for synchronizing a set of coupled harmonic oscillators by using some sampled data that contains measurement noise. We have showed that the oscillators can be synchronized even when the velocity information is exchanged only at discrete-time instants and discussed the convergence of the algorithm for directed networks with either fixed or switching topologies in the presence or absence of leaders. We have also established the sufficient conditions for which the coupled harmonic oscillators could attain synchronized oscillatory motions containing noise. Some numerical examples are given to illustrate the effectiveness of the theoretical results.

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