Lebesgue Approximation Model of Continuous-Time Nonlinear Dynamic Systems

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Abstract

Traditional model-based approaches are based on periodic iterations, where the continuous-time model is discretized with a fixed period. Despite the easiness in analysis and design, such periodic approximation model may be undesirable from the computation-efficiency point of view. This paper presents the Lebesgue-approximation model (LAM) of continuous-time nonlinear systems, where the iteration is activated on an “as-needed” basis, but not periodically. We show that the proposed LAM behaves exactly the same as a specific event-triggered feedback system, through which the properties of the LAM can be studied. We provide a sufficient condition to ensure asymptotic stability of the LAM and derive theoretical bounds on the difference between the states of the LAM and the original continuous-time system. The LAM is then integrated in the particle-filtering approach for fault prognosis. Simulation results show that the LAM can dramatically reduce the number of iterations in prognosis without sacrificing accuracy and precision.

Key words: System approximation, Lebesgue measurement, sporadic model, model-based, fault prognosis

1 Introduction

Model-based approaches are widely used in control community. In such approaches, the physical dynamics can be described by a mathematical model, based on which one can address problems associated with controller synthesis and analysis, optimization, and prediction, etc. When implementing model-based algorithms in computers, the traditional method is to approximate the model with a fixed period over the time horizon and iterations take place periodically. Such a periodic approximation model is known as “Riemann Approximation model” (RAM) [2].

A nice feature of using RAM is the easiness in analysis and design. However, it may be undesirable in many situations, from the computation-efficiency point of view. On the one hand, since the iteration period is determined according to the worst-case operating scenario, the state of the model might be iterated even if there is little change in the actual states. In other words, it may take greater computational resources than it actually needs. This may result in significant over-provisioning of the real-time system hardware, in particular for embedded processors which have very limited computation

ability [11, 12]. On the other hand, when the system becomes unstable, the state will diverge exponentially. In this case, it is expected to perform iterations more frequently so that the model can closely track the actual state, which cannot be met by RAM. It suggests cost-efficient approaches to construct the approximation models where iterations can be executed on an “as-needed” basis.

One possible solution is the discrete event simulation (DEVS) formalism, where the concept of the quantized state system (QSS) was introduced for real-time simulation [14]. The basic idea of QSS is to first quantize the state space based on a pre-defined quantizer $Q(\cdot)$ and then define transitions between the pre-defined quantized states characterized by $Q(\cdot)$. The frequency of state transition really depends on the state of the model, but not directly on the fixed amount of time elapsed. The existence of stabilizing quantizers has been established in [6, 7, 14]. This approach was further explored in [4, 5, 7], which extends the original approach to illegitimate models. These results indicate that the DEVS promises a significant improvement of computational efficiency in real-time simulations of large and complex systems such as power systems [8]. Although QSS demonstrates great advantages in cost-efficiency, there is a lack of theoretical foundation and design methods to construct stabilizing quantizers. It is also worth mentioning that all of the prior work on QSS focuses on the uniform quantizer.

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Inspired by the QSS approach, this paper considers sporadic discretized model of dynamic systems. Different from the work in [9, 13] using stochastic approaches, the model developed in this paper is completely deterministic. Another approach similar in spirit is Lebesgue piecewise affine approximation in [3], where the state space is partitioned into a number of sets and the nonlinear dynamics can be approximated by a linear system over each set. However, the state space is partitioned independent of the initial state in [3] and the nonlinearity has to take a special form $f(x) = \sum_{k=1}^N \prod_{i=1}^n f_{ik}(x_i)$ where x_i is the i th element of x . Moreover, the time information is not explicitly characterized in this approach.

This paper considers sporadic approximation models of multi-dimension continuous-time nonlinear systems, including aperiodic iterations in both state and time. We study stability of this model and its closeness to the continuous-time system. The contributions of this paper are as follows: (i) We present the Lebesgue approximation model (LAM), where the next state is computed only when it is certain distance away from the previous state. Different from the QSS, the states in the LAM do not have to be pre-defined. Another significant difference is that the quantizer in the LAM does not have to be uniform. Compared with [3], the initial state and iterations in time are taken into account in the state space partition; (ii) We develop sufficient conditions to ensure asymptotic stability of the LAM without exhibits of Zeno behavior. Theoretical bounds are derived to quantify the difference between the states of the LAM and the continuous-time system; and (iii) We apply the LAM in model-based fault prognosis. Experiments show that the LAM-based prognosis can dramatically reduce the computational costs without sacrificing accuracy.

2 Problem Formulation

Notations: We denote by $\mathbb{R}^n$ the n -dimensional real vector space, by $\mathbb{R}^+$ the set of the real positive numbers. We use $\|\cdot\|$ to denote the Euclidean norm of a vector and the induced 2-norm of a matrix. The symbol e is used for exponential function to distinguish it from the error e . Given two functions $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$, we define $\phi \circ \psi(t) = \phi(\psi(t))$. Given a differentiable function $f(x)$, its gradient is denoted by $\nabla f(x)$. For a function of time $x(t)$, sometimes we drop t for brevity if it is clear in context.

Consider an autonomous system:

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0 \quad (1)$$

where $x : \mathbb{R} \rightarrow \mathbb{R}^n$ is the system state, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a known function, and $x_0 \in \mathbb{R}^n$ is the initial state. In general, the discretized model of this continuous-time system can be described by the iterative equations:

$$\hat{x}(t_{k+1}) = \hat{f}(\hat{x}(t_k)) \quad (2a)$$

$$\hat{x}(t_0) = x_0 \quad (2b)$$

$$t_{k+1} = t_k + \hat{g}(\hat{x}(t_k)) \quad (2c)$$

where the functions $\hat{f}$ and $\hat{g}$ describe the iteration in the state and the time, respectively, which are to be determined. The discrete-time signal $\hat{x}(t_k)$ is the state of this discrete-time model at t_k . Starting from $\hat{x}(t_0)$, this model will generate two sequences: the time sequence $\{t_k\}_{k=0}^\infty$ and the state sequence $\{\hat{x}(t_k)\}_{k=0}^\infty$.

Once the discrete-time model is established, we can use different interpolation methods to construct the states over (t_k, t_{k+1}) such as $\hat{x}(t) = h(\hat{x}(t_k), \hat{x}(t_{k+1}), t)$ for any $t \in (t_k, t_{k+1})$. Thus, the discrete-time model (2) and the interpolation generate a continuous-time trajectory $\hat{x}(t)$, which is expected to approximate the actual state $x(t)$ of the continuous-time system in (1).

The objective is to construct sporadic real-time models in the form of (2) and develop guidelines on the quantization design that ensure stability of the resulting model as well as its closeness to the continuous-time system in (1), with the cost-efficient purpose.

Remark 2.1 Notice that the LAM is completely different from the self-triggered feedback scheme [1, 12]. Self-triggered control only needs the iteration equation on time (equation (2c)), while the states are always sampled from the plant. The signal $\hat{x}(t_k)$ in the LAM is not the sampled state, but the predicted state that is completely based on the model itself.

3 Lebesgue Approximation Model

The basic idea of the LAM is to have iterations executed only when the state is “significantly” different from the previous one. The “significance” is measured by the threshold $D(x)$, which is also referred to the quantization size [7]. The LAM is described as follows:

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + D(\hat{x}(t_k)) \cdot \frac{f(\hat{x}(t_k))}{\|f(\hat{x}(t_k))\|} \quad (3a)$$

$$\hat{x}(t_0) = x_0 \quad (3b)$$

$$t_{k+1} = t_k + \frac{D(\hat{x}(t_k))}{\|f(\hat{x}(t_k))\|} \quad (3c)$$

where $\hat{x}(t_k)$ is called the Lebesgue state and $D : \mathbb{R}^n \rightarrow \mathbb{R}^+$ represents the quantization size.

We consider linear interpolation to construct the states between $\hat{x}(t_k)$ and $\hat{x}(t_{k+1})$: for any $t \in (t_k, t_{k+1})$,

$$\hat{x}(t) = \hat{x}(t_k) + f(\hat{x}(t_k)) \cdot (t - t_k). \quad (4)$$

Figure 1 shows the state trajectories generated by a scalar continuous-time system (“solid”) and the related LAM (“dashed”) with uniform quantization size D . Assume that at t_k , $\hat{x}(t_k) = x(t_k)$. The next state $\hat{x}(t_{k+1})$ is simply $\hat{x}(t_k) + D$. The inter-transition time $t_{k+1} - t_k$ is based on the slope at $\hat{x}(t_k)$. This iteration will generate a gap between $\hat{x}(t_{k+1})$ and $x(t_{k+1})$ and this gap will

be propagated into the next iterations, which may possibly be amplified. Thus, it is important to study stability of the LAM and how close it is compared with the continuous-time system.

Fig. 1. The state trajectories generated by the LAM and the continuous-time system

4 Equivalence to Event-Triggered Systems

To show the properties of the LAM, it might be difficult to directly analyse the model by itself. Alternatively, we build up the connection between the LAM and a specific event-triggered feedback system. We show that the state trajectories of these two systems are identical. Therefore, the properties of the LAM can be explored through examining the event-triggered system, for which many analysis tools have been developed [11].

Consider the event-triggered feedback system, whose plant is as follows:

$$\dot{z}(t) = v(t), \quad z(t_0) = x_0 \quad (5)$$

where $z : \mathbb{R} \rightarrow \mathbb{R}^n$ is the state of the system and $v : \mathbb{R} \rightarrow \mathbb{R}^n$ is the input. The event is generated when the logic

$$\|z(t) - z(\tau_k)\| \leq D(z(\tau_k)) \quad (6)$$

is violated. Thus, the next sampling time is defined by

$$\tau_{k+1} = \inf_{t \geq \tau_k} \{t \mid \|z(t) - z(\tau_k)\| > D(z(\tau_k))\} \quad (7)$$

with $\tau_0 = t_0$. This triggering condition ensures that inequality (6) always holds. When the event takes place, the system samples the state and passes the sampled state to the controller to compute the new control input with the control law

$$v(t) = f(z(\tau_k)), \quad \forall t \in [\tau_k, \tau_{k+1}). \quad (8)$$

The following theorem establishes the equivalence between the LAM and the event-triggered system.

Theorem 1 *The state trajectories of the LAM (3) and the event-triggered feedback system (5) – (8) are identical, i.e. $z(t) \equiv \hat{x}(t)$ for any $t \geq t_0$ and $\tau_k \equiv t_k$ for all k .*

PROOF. The proof is straightforward and therefore omitted due to space limitation. ■

5 Stability and Closeness

This section studies the stability of the LAM and its deviation from the actual system (1) through comparing the system (1) and the event-triggered system in (5) – (8). We replace the time index τ_k in the event-triggered system by t_k , since they are identical by Theorem 1.

Assumption 5.1 *The continuous-time autonomous system in (1) is stable and admits a Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$, such that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ and $\frac{\partial V(x)}{\partial x} f(x) \leq -\alpha_3(\|x\|)$, where $\alpha_i, i = 1, 2, 3$ are class $\mathcal{K}$ functions. Moreover, $\frac{\partial V(x)}{\partial x}$ is locally Lipschitz with $L_{\partial V}$ as the Lipschitz constant over Ω where*

$$\Omega \triangleq \{x \in \mathbb{R}^n \mid V(x) \leq V(x_0)\}. \quad (9)$$

Proposition 5.1 (Stability) *Suppose that Assumption 5.1 holds. If there exists a positive constant $\rho \in (0, 1)$ such that the quantization size $D(x)$ satisfies $D(x) \leq \frac{\rho}{L_{\partial V}} \frac{\alpha_3(\|x\|)}{\|f(x)\|}$ for any $x \in \Omega$, then $\hat{x}(t) \in \Omega$ for $\forall t \geq t_0$ and the LAM is asymptotically stable, i.e. $\lim_{t \rightarrow \infty} \hat{x}(t) = 0$. Moreover, if $D(x) \geq \beta(\|x\|)$ and $\lim_{x \rightarrow 0} \frac{D(x)}{\|f(x)\|} > 0$, where $\beta(\|x\|)$ is a class $\mathcal{K}$ function, the inter-transition intervals $t_{k+1} - t_k$ will be strictly bounded from below.*

PROOF. The proof is similar to that of Theorem 3.1 in [11] and therefore omitted. ■

Proposition 5.1 provides the criteria to design quantizers that can guarantee asymptotic stability of the LAM and avoid Zeno behavior [11]. Next, we examine the deviation of the LAM from the continuous-time system.

Proposition 5.2 (Closeness) *Suppose that Assumption 5.1 holds and there exist $c_1, c_2, c_3 \in \mathbb{R}^+$ such that*

$$\alpha_1(\|x\|) \geq c_1\|x\|^2, \alpha_2(\|x\|) \leq c_2\|x\|^2, \alpha_3(\|x\|) \geq c_3\|x\|^2.$$

Also assume that $f(x)$ and $\nabla f(x)$ are locally Lipschitz with L_f and $L_{\nabla f}$ as the Lipschitz constants over Ω , respectively. Given a positive constant η satisfying

$$3c_2L_{\nabla f}L_{\partial V}\eta < c_1c_3. \quad (10)$$

If $d_{\max} = \max_{x \in \Omega} D(x)$ satisfies

$$d_{\max} < \frac{\frac{c_1c_3}{c_2L_{\partial V}} - (3\eta + \epsilon)L_{\nabla f}}{\sigma(L_{\nabla f}d_{\max} + L_{\nabla f}\alpha_1^{-1} \circ V(x_0) + \|\nabla f(0)\|)}, \quad (11)$$

where $\epsilon \in \mathbb{R}^+$ satisfies

$$\epsilon < \frac{c_1c_3}{c_2L_{\nabla f}L_{\partial V}} - 3\eta, \quad (12)$$

$T \in \mathbb{R}^+$ is the time instant when $x(t)$ enters the set $\{x \in \mathbb{R}^+ \mid \|x\| \leq \epsilon\}$ after t_0 , and $\sigma = e^{\frac{L_{\partial V} L_{\nabla f} \alpha_1^{-1} \circ V(x_0) T}{c_1}}$, then $\|x(t) - \hat{x}(t)\| \leq \eta$ holds for any $t \geq t_0$.

Remark 5.1 A stable LAM can be applied to applications that seek stability. An example is its application in model predictive control (MPC). It can be used in MPC to reduce the amount of computation in solving the finite-horizon optimal control problem at each iteration. We will investigate this idea in the future.

Example. Consider a continuous-time stable system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -4(x_1 + x_2) - 0.1(x_1 + x_2)^3$$

with the initial condition $x(0)$ randomly selected over $[-0.1, 0.1] \times [-0.1, 0.1]$. The Lyapunov function is

$$V(x) = 2x_1^2 + 2x_1x_2 + x_2^2.$$

One can verify that over the set Ω

$$\begin{aligned} \alpha_1(\|x\|) &= 0.382\|x\|^2, & \alpha_2(\|x\|) &= 2.618\|x\|^2 \\ \alpha_3(\|x\|) &= 0.917\|x\|^2, & L_{\partial V} &= 5.236, \\ L_f &= 6, & L_{\nabla f} &= 1.2 \end{aligned}$$

By choosing the dynamic quantizer $D(x) = 0.029\|x\|$, the conditions in Proposition 5.1 can be verified, which implies the resulting LAM will also be stable. The generated state trajectory is almost identical to that of the continuous-time system. However, if only focusing on stability of the LAM, $D(x)$ can be as large as $1.5\|x\|$ in the simulation, although the resulting state trajectory significantly deviates from the actual one.

To verify the conditions in Proposition 5.2, we consider a constant $D(x) = 6 \times 10^{-5}$ and the resulting error bound is $\eta = 7.1 \times 10^{-4}$. Obviously, it is conservative due to the over-approximation for nonlinear systems. In simulation the actual error bound is 3×10^{-6} with the same $D(x)$. $\square$

When the continuous-time system is unstable, the closeness of the LAM has to be discussed over a finite horizon. The following proposition quantifies the growth of the gap between states of the LAM and the unstable system.

Proposition 5.3 Assume that $f(x)$ is globally Lipschitz and $D(x) \equiv d$ with a constant d . Then $\|x(t) - \hat{x}(t)\| \leq d(e^{L_f(t-t_0)} - 1)$. Moreover, if there are $t^* \geq t_0$ and a positive constant c such that $\|x(t)\| \geq c$ for any $t \geq t^*$, then for any $t \geq t^*$, the normalized error satisfies

$$\frac{\|x(t) - \hat{x}(t)\|}{\|x(t)\|} \leq \frac{d}{2c} \left(2e^{L_f(2t-t^*-t_0)} - e^{2L_f(t-t^*)} - 1 \right).$$

PROOF. The proof follows the idea of Theorem 6.3 in [12] and therefore is omitted. $\blacksquare$

6 Application to LAM-Based Fault Prognosis

This section presents the experimental results on LAM-based prognosis on a five-planet epicycle gear system [15], which provides power to the main transmission of helicopters. The fault of interest is a crack in the planetary carrier plate that may result in major overhaul and replacement of gear plates with high cost.

In the experiment a fault is seeded on the carrier plate and grew with the evolving operation of the gearbox. The gearbox with the seeded fault operated over a large number of ground-air-ground (GAG) cycles and each GAG cycle is about 3 min. The feature values, as shown in Figure 2, are extracted from the vibration signals from GAG cycles to indicate the crack growth [15].

Fig. 2. Feature values with crack growth

A simplified model for the fatigue crack growth based on Paris's law is given as follows:

$$\frac{dL}{dn} = C(L)^m + \omega. \quad (13)$$

where L is the crack length, C and m are material related coefficients, n is cycle index, and ω is the noise. The fault is a seeded crack. Notice that the crack length cannot be directly measured. So the feature values are used to indicate the crack length in this experiment based on an assumption of one-to-one mapping between them, which is over [1.28 4.57]. The probability of false alarm is set as 5% and the confidence of detection is set as 95%. The failure threshold for the crack length, L_{fail} , is set as 4 inches for prognosis to calculate the time-to-failure (TTF) or the remaining useful life (RUL). Since the fault growth is a nonlinear and non-Gaussian process, analytical solution may not exist. We use the particle filtering approach [10] for prognosis where the prediction is based upon LAM.

Using the diagnostic algorithm proposed in [15], a fault is detected at the 183th cycle. This detection triggers the *prognostic task* that predicts the TTF or RUL. Notice that the prediction horizon is over the axis of the state but not the time axis. So the initial condition of prognosis is a time distribution for the fault to reach the current state L_{ini} . This initial time distribution can be obtained by predicting the time when those particles

(that have not reached L_{ini} yet) will arrive at L_{ini} , using the law of total probabilities. It can be mathematically described as:

$$p_{ini}^t = \sum_{i=1}^N \Pr(L_i^t > L_{ini}) w_i^t$$

where superscript i is the index of particles, p_{ini}^t is the probability of reaching L_{ini} at time t , N is the number of particles, w_i^t is the weight of particles that have not reached L_{ini} , and L_i^t is the predicted value of the i th particle. The probability $\Pr(L_i^t > L_{ini})$ is obtained based on the predicted probability density function (pdf) of the state at each circle. The output of this is a time distribution of the fault state to reach the state L_{ini} that serves as the initial condition for LAM-based prognosis.

Starting from L_{ini} , the LAM can be established for prognosis with a constant quantization size D . The LAM-based prognosis calculates the distribution of the operation time to reach each Lebesgue state and the prognostic horizon is the number of Lebesgue states from the current state L_{ini} to the one that indicates the failure threshold L_{fail} . Using equation (3a), totally 15 Lebesgue states are generated. This number is small compare to that in RAM-based prognosis, which is about 600 cycles. It significantly reduces the computation effort and enables the LAM-based prognosis to use 500 particles, which are too many for RAM-based prognosis [10].

Recall that the Lebesgue states themselves do not contain the information on time. To predict the TTF or RUL, we need the associated time information which is provided by equation (3c). The prognosis results at the 183th cycle are shown in Figure 3. To make the figure clear, only the time distribution pdf at a few selected Lebesgue state are plotted. Notice that the time distribution pdf at the Lebesgue state defined by the failure threshold gives the RUL estimation pdf. In this figure, the predicted failure time is at the 689.4 cycle and the RUL life is 503.6 cycle. The 95% confidence bound of the RUL pdf is given as [601 747.6].

Fig. 3. Experimental result of LAM-based prognosis. The advantages of LAM in fault prognosis are obvious when the results of RAM-based and LAM-based prog-

nosis are compared. The comparison are summarized in Table I. First of all, LAM-based prognosis can afford more particles than RAM-based prognosis (when the number of particles exceeds 20, the computation in RAM-based prognosis becomes very slow). Secondly, the LAM-based prognosis is 353.1 times ($1.928/0.00546$) faster than RAM-based prognosis even when using 500 particles. Third, the LAM-based prognosis shows better performance in terms of accuracy and precision. The 95% confidence interval (CI) of TTF distribution for RAM-based prognosis is 324 cycles, while that of LAM-based prognosis is only 146.6 cycles. Moreover, the TTF expectation is 689.4 cycles, which is much closer to the actual TTF (750 cycles).

Table I: Comparison of Prognosis Using RAM and LAM

	RAM	LAM
# of Particles	20	500
TTF Expectation	588.6	689.4
TTF 95% CI	[443, 767]	[601, 747.6]
Prognosis Horizon	605	15
Computation Time	1.928	0.00546

7 Conclusions

This paper presents the LAM for continuous-time nonlinear systems. Sufficient conditions are provided to ensure stability and closeness of the LAM. With the cost-efficient nature, LAM could be a promising technique to implement model-based algorithms in systems with computation limitations. Interesting problems include how to use such models to address the problems associated with estimation, prediction, and control, as well as the development of LAMs for different types of systems.

8 Sketch of the Proof of Proposition 5.2

Since $\hat{x}(t) = z(t)$ by Theorem 1, the state error between the actual system in (1) and its LAM (3) can be studied by examining the error dynamics between the actual system and the event-triggered system in (5) – (8). The error dynamics are $\dot{e} = f(x) - f(\hat{z}) = f(e) + \Delta(e)$, with the initial condition $e(t_0) = 0$, where $e(t) = x(t) - z(t)$, $\hat{z}(t) = z(t_k)$ for any $t \in [t_k, t_{k+1})$, and $\Delta(e) = f(x) - f(\hat{z}) - f(e)$. Given $f(0) = 0$, by the mean value theorem, the i th entry of Δ can be rewritten as

$$\begin{aligned} \Delta_i(e) = & (\nabla f_m^i(\lambda_{i1}(\hat{z} - x) + x) - \nabla f_m^i(0)) \cdot (x - \hat{z}) \\ & - (\nabla f^i(\lambda_{i2}e) - \nabla f_m^i(0)) \cdot e \\ & + (\nabla f_m^i(0)(x - \hat{z}) - \nabla f_m^i(0)e) \end{aligned}$$

where $f^i(x)$ is the i th entry of $f(x)$ and $\lambda_{ij} \in (0, 1)$ for $j = 1, 2$. Since $\frac{\partial f(x)}{\partial x}$ is locally Lipschitz, we have

$$\begin{aligned} \|\Delta(e)\| \leq & L_{\nabla f} (\|x - \hat{z}\| + \|x\|) \|x - \hat{z}\| \\ & + L_{\nabla f} \|e\|^2 + \|\nabla f(0)\tilde{z}\|. \end{aligned}$$

Note that $\tilde{z} = x - \hat{z} - e$. Consider the time derivative of the Lyapunov function $V(e)$. According to Assumption 5.1, together with the inequality above, we have

$$\dot{V} \leq -c_3 \|e\|^2 + L_{\partial V} \|e\| (L_{\nabla f} (\|x - \hat{z}\| + \|x\|) \|x - \hat{z}\| + L_{\partial V} \|e\| (L_{\nabla f} \|e\|^2 + \|\nabla f(0)\tilde{z}\|).$$

Let $\tilde{z} = z - \hat{z}$. Then $x - \hat{z} = z + e - \hat{z} = \tilde{z} + e$. Applying these two equations to the preceding inequality to replace $x - \hat{z}$ and $\tilde{z}$, we have

$$\dot{V} \leq -c_3 \|e\|^2 + L_{\partial V} \|e\| \|\nabla f(0)\tilde{z}\| + L_{\partial V} L_{\nabla f} \|e\|^3 + L_{\partial V} L_{\nabla f} \|e\| (2\|\tilde{z}\|^2 + 2\|e\|^2 + \|x\|(\|\tilde{z}\| + \|e\|)).$$

Notice that in the event-triggered system, $\|\tilde{z}\| \leq D(\hat{z})$ holds. Since $\hat{z}(t) = \hat{x}(t) \in \Omega$, we have $D(\hat{z}) \leq d_{\max}$. Therefore, the inequality above can be further derived:

$$\dot{V} \leq -\frac{c_3}{c_2} V(e) + \frac{L_{\partial V} L_{\nabla f} (3\|e\| + \|x\|)}{c_1} V(e) + d_{\max} L_{\partial V} (L_{\nabla f} d_{\max} + L_{\nabla f} \|x\| + \|\nabla f(0)\|) \|e\|. \quad (14)$$

We now prove $\|x(t) - \hat{x}(t)\| \leq \eta$ by contradiction. Suppose that the inequality is not true. Since $e(t)$ is continuous and $\|e(t_0)\| = 0 < \eta$, there must exist $\tau^* > t_0$ so that $\|e(\tau^*)\| = \eta$ and $\|e(t)\| < \eta$ for any $t \in [t_0, \tau^*)$. Let

$$\kappa(t) = \frac{L_{\partial V} L_{\nabla f} (3\eta + \|x\|)}{c_1} \quad (15)$$

$$\xi = \eta d_{\max} L_{\partial V} (L_{\nabla f} d_{\max} + L_{\nabla f} \alpha_1^{-1} \circ V(x_0)) + \eta d_{\max} L_{\partial V} \|\nabla f(0)\|. \quad (16)$$

Thus, inequality (14) implies $\dot{V} \leq -\left(\frac{c_3}{c_2} - \kappa(t)\right) V(e) + \xi$ for any $t \in [t_0, \tau^*)$, because $x(t) \in \Omega$ by Assumption 5.1 and $\|e(t)\| \leq \eta$ over $[t_0, \tau^*)$. Since $e(t_0) = 0$, we have $V(t_0) = 0$. Then solving the preceding differential inequality implies

$$V(e) \leq \int_{t_0}^t \varphi(t - t_0, \tau) \xi d\tau, \quad (17)$$

for all $t \in [t_0, \tau^*)$, where

$$\varphi(t, \tau) = e^{-\frac{c_3}{c_2}(t-\tau) + \int_{\tau}^t \kappa(s) ds}. \quad (18)$$

Consider $\kappa(t)$ in equation (15). By Assumption 5.1, $\lim_{t \rightarrow 0} x(t) = 0$. Then there must exist a small enough $\epsilon > 0$ and a time instant $T > 0$, such that (i) $\|x(t)\| \leq \epsilon$, $\forall t \geq T + t_0$, (ii) $\|x(t)\| \leq \alpha_1^{-1} \circ V(x_0)$, $\forall t \geq t_0$, and (iii) $\frac{c_3}{c_2} > \frac{L_{\nabla f} L_{\partial V} (3\eta + \epsilon)}{c_1} = \mu$, where the last inequality is satisfied because of inequality (12). Thus, $\int_{\tau}^t \|x(s)\| ds \leq \epsilon(t - \tau) + \alpha_1^{-1} \circ V(x_0)T$ holds for any $t \geq t_0$. Furthermore, it implies that $\int_{\tau}^t \kappa(s) ds \leq \mu(t - \tau) + \frac{L_{\partial V} L_{\nabla f} \alpha_1^{-1} \circ V(x_0)T}{c_1}$ for any

$t \in [t_0, \tau^*)$. Applying this inequality back into the definition of $\varphi(t, \tau)$ in (18) yields $\varphi(t, \tau) \leq e^{-\sigma_1(t-\tau)\sigma}$, where $\sigma_1 = \frac{c_3}{c_2} - \mu$. Using this inequality in (17) implies $V(e) \leq \int_{t_0}^t e^{-\sigma_1(t-t_0-\tau)} \sigma_2 \xi d\tau \leq \frac{\sigma_2 \xi}{\sigma_1}$. Thus, for any $t \in [t_0, \tau^*)$, $\|e(t)\| \leq \frac{\sigma_2 \xi}{\sigma_1 c_1} < \eta$, where the inequality at the right side comes from inequality (11). This is contradicted with the assumption that $\|e(\tau^*)\| = \eta$. Therefore, we conclude that $\|e(t)\| \leq \eta$ for any $t \geq t_0$. ■

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