Identifiability and Excitation of Linearly Parametrized Rational Systems*

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Abstract—This paper establishes identifiability and informativity conditions for a class of deterministic linearly parametrized scalar rational models. The class considered is rational in the state and polynomial in the input. The standard definitions of identifiability and informativity for linear systems are expanded to account for the situation, common for nonlinear deterministic systems, where the identification is achieved either through the application of an informative input or via the response to an informative initial condition. We provide necessary and sufficient conditions for identifiability from the initial state, respectively from the input, as well as necessary and sufficient conditions on the initial state to produce an informative experiment. We also provide sufficient conditions on the input to be informative when the initial condition is unknown and could therefore potentially destroy the transfer of information from the input to the regressor.

I. INTRODUCTION

The question of identifiability of parametrized dynamical systems has occupied generations of system theorists, and the very definition of this concept has evolved over the years. For a long time, this concept embraced both the parametrization issue and the richness of the data set. Eventually, a clear separation was made between the *identifiability of the model structure*, which is a parametrization issue, and the *informativity of the data*, which is the issue of applying an excitation to the system that will produce different responses for different parameter values.

The question of **identifiability of the model structure** can be succinctly summarized as follows: does there exist an experiment such that the data collected from this experiment allow one to uniquely determine the parameter values? From a more analytical point of view this question translates into the following: *is the mapping from parameter vector θ to model $M(\theta)$ injective?*

It is important to observe that identifiability is a property of the chosen model structure (i.e. the parametrization); it is totally independent of the true system and of the data.

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The seminal papers [8] and [16] provided a broad answer to this identifiability question for large classes of linearly and nonlinearly parametrized systems using tools from differential algebra. In [8] it is shown that a model class is identifiable if it is equivalent to a linear regression. Checking this equivalence condition for a given model class is by no means a trivial matter, and a solution for this problem is provided in [16], based on the application of Ritt's algorithm to the model equations. Ritt's algorithm essentially produces a set of differential algebraic equations that are equivalent to the initial model structure, by a long sequence of differentiations and substitutions. At the end of the procedure, sufficient conditions for global identifiability are obtained if the final set of equations consists of linear regressions; the satisfaction of these conditions hinges on whether one can produce an experiment (i.e. an input signal) that makes a number of polynomials in u , y , and their derivatives nonzero. This is the informativity question.

While Ritt's algorithm is very powerful, it is also very computationally intensive. Even for rather simple-looking model structures, it may be impossible to terminate the computations of Ritt's algorithm on a standard PC. In addition, except for some very simple examples, the work of [16] and subsequent work on identifiability and informativity of nonlinear systems does not address the question of how to produce an informative input signal. Hence the desire to produce alternative and more efficient methods for special classes of nonlinear systems that better reveal the connection between identifiability, informativity and the parametric structure of the model class. This is the goal of the present paper where we address a class of linearly parametrized rational systems.

The question of **informativity of the data** turns out to be much harder to solve than that of identifiability. For linear time invariant (LTI) systems, sufficient conditions on the input had been available for a long time (see e.g. [15]), but necessary and sufficient conditions remained elusive until this question was solved in [12]. The relationship between identifiability, informativity and the uniqueness of the minimum in a Prediction Error Identification framework was established in [2] and required the introduction of the new concept of *local informativity*.

Informativity of the data - also known as input richness or transfer of excitation - is a topic that has attracted and continues to attract a wide attention: it is an experiment design issue. It is important not just in the context of identification, but also for the convergence of adaptive estimation and control schemes. Simply stated, the question can

be summarized as follows: how to choose an experiment such that the data collected from this experiment suffices to uniquely determine the parameter values, once it has been determined that such an experiment exists (that is, once it has been determined that the model structure is identifiable)? Again, on a more analytical note, this question translates into: *What are the conditions on the excitation signals that will make the Gramian associated with a certain regression vector full rank?* When that Gramian is full rank, the regressor vector is said to be informative. The full rank condition on the Gramian is required for the estimation of the parameters. The excitation signal is typically an input signal, but in this paper we examine the situation where it can also be a properly chosen initial condition or a combination of both.

Besides the results for the identification of LTI systems mentioned above, a wide range of questions related to informativity of the data and transfer of excitation have been addressed, dealing with different classes of systems, different types of input, and different convergence requirements.

In [1] informativity conditions on the input have been obtained for parameter convergence in linear discrete-time adaptive control schemes. Similar conditions for continuous-time linear adaptive control systems have been derived in [13]. In [14] nonlinear adaptive control schemes were studied, and it was shown that nonlinearities may reduce the requirements on the richness of the external signal.

In the context of system identification, informativity conditions have been obtained for linear time-varying systems in [18], and more specific results have been obtained in [5] for Linear Parameter Varying systems with an ARX structure. In [7] informativity conditions on the input signal have been derived for a class of discrete-time linearly parametrized systems that are linear in the output and polynomial in the input. Bilinear systems are special members of this class. In [20] the question of which type of input signals (e.g. pulses, impulses, etc) are sufficient for the identification of bilinear systems has been studied.

In [6] necessary and sufficient conditions were derived for the identifiability of continuous time scalar linearly parametrized deterministic polynomial systems. The harder question of how to generate informative inputs for such systems was raised in that paper but not solved. In addition, identification based on the response to an initial condition was not considered in [6]. A first step towards addressing these issues was presented in a recent paper [11], where results on identifiability and informativity for scalar linearly parametrized systems that are polynomial (rather than rational) in the state were presented.

This paper expands the results of earlier work on identifiability and informativity in several different ways. This is made possible by considering linearly parametrized model classes with a scalar state. Extensions to vector states of some results given in this paper have recently been obtained [3].

A first major extension is that we derive identifiability and informativity conditions for two practically important special

cases: the first is when the system must be identifiable from the response to an initial condition only, i.e. for a system that has no inputs; the second is when the identifiability must be secured from the response to an informative input without any knowledge of the initial condition, i.e. one must guarantee that an “adversarial” initial condition does not kill the transfer of excitation from input to state. So far, most results on identifiability quoted above have considered identifiability from the input only. It is common in some application fields that the parameters are identified from data obtained as the response to some initial condition without any external input.

For the class of scalar linearly parametrized rational systems studied in this paper, we derive necessary and sufficient conditions for identifiability from either the initial state, or from the input with unknown initial state; these conditions are on the structure of the model class only. For the case of identification from the initial state, we present necessary and sufficient conditions on this initial state to be informative. Finally, we present sufficient conditions on the input to be informative, in the context where the initial state is unknown and may therefore potentially destroy the transfer of excitation from the input to the regressor.

The remainder of the paper is organized as follows. In Section II we present the nonlinear model class that is treated in this paper and the fundamental assumptions on the signals and the models. In Section III the definitions of identifiability and informativity are extended from the linear stationary stochastic case to the case of nonlinear deterministic systems. Fundamental results on identifiability for the considered class of model structures are given in Section IV. From that point on, we depart from the classical setting in which identifiability and informativity are secured from the input, to definitions that apply when these can also be secured from the response to an initial condition. The tools developed in Section IV are essential in the subsequent developments. In Section V we study the identifiability and informativity from the initial state with zero input, providing the formal definitions of these concepts and necessary and sufficient conditions for both. Then we approach, in Section VI, the problem of identifiability and informativity from the input, regardless of the initial state. Again the concepts are formalized and conditions for identifiability and informativity are presented. An example is given that illustrates the difficulty of obtaining necessary and sufficient conditions on the input to be informative regardless of the initial state. Section VII concludes the paper.

II. MODEL CLASS AND ASSUMPTIONS

Consider the following class of deterministic continuous-time nonlinear model structures:

$$\dot{x} = f(x, \theta) + g(x, u), \quad x(0) = x_0 \quad (1)$$

where $x : \mathbb{R} \rightarrow \mathbb{R}$ is a scalar state, $\theta \in \mathbb{R}^d$ is an unknown vector, $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are given (i.e. known) scalar analytical functions and $g(\cdot, 0) = 0$. We shall specialize in this paper to a class of linearly parametrized models that are

rational in the state and polynomial in the input, with the following form for $f(x, \theta)$ and $g(x, u)$:

$$f(x, \theta) = \frac{1}{n(x)}[\theta^T \phi(x) + m(x)] \quad (2)$$

$$g(x, u) = \frac{1}{n(x)} \sum_{i=1}^l g_i(x) u^i = \frac{1}{n(x)} G(x) U \quad (3)$$

where $n(x)$ is a known polynomial such that $n(x) > 0 \forall x \in \mathbb{R}$, $\phi(x) \in \mathbb{R}^d$ is a known polynomial vector in the scalar x , $m(x)$ is a known polynomial in x , $g_i(x) \in \mathbb{R}$ are known polynomials in x , and where $G(x)$ and U are defined as follows:

$$G(x) = [g_1(x) \ g_2(x) \ \dots \ g_l(x)], \ U = [u \ u^2 \ \dots \ u^l]^T \quad (4)$$

We denote by q the polynomial degree of $\phi(x)$, i.e. the degree of the highest degree polynomial in $\phi(x)$. In order to avoid a degenerate situation, we make the following assumption concerning the model structure.

Assumption 1: The term $m(x)$ in (2) is such that $\nexists \eta \neq 0 : m(x) \equiv \eta^T \phi(x)$.

This assumption does not represent any loss of generality: if $m(x) \equiv \eta^T \phi(x)$ for some η , then the model could be rewritten as $f(x, \rho) = \frac{1}{n(x)}[\rho^T \phi(x)]$ with the new parameter vector $\rho = \theta + \eta$. The family of all models (1) generated by all $\theta \in \mathbb{R}^d$ is called the model class $\mathcal{M}$.

Concerning the signals, we shall make the following assumption which is standard in nonlinear system identification; see, e.g., [6] and [16].

Assumption 2: The signal $u(t)$ is analytic and the solution $x(t)$ of (1) is an analytic function of time for all $u(t)$ and $x(0)$.

The virtue of the above assumption is that knowing all derivatives of an analytic signal at some time is equivalent to knowing that signal at all times.

Ideally the choice of parameterization made in (1) should be such that the model class can describe exactly the true system; we shall throughout make the following assumption.

Assumption 3: There exists a parameter value $\theta_0 \neq 0$ such that the true system is described by (1) with $\theta = \theta_0$.

These three assumptions will be in force throughout the paper and will not be repeated in our theorem statements.

The model structure (1) can be rewritten as

$$\dot{x} = \frac{\theta^T \phi(x) + m(x) + G(x)U}{n(x)}, \quad x(0) = x_0 \quad (5)$$

or, alternatively, as

$$n(x)\dot{x} = \theta^T \phi(x) + m(x) + G(x)U, \quad x(0) = x_0 \quad (6)$$

This latter expression will be most convenient for some of the derivations to follow.

III. THE CONCEPTS OF IDENTIFIABILITY AND INFORMATIVITY

We now present definitions that are the nonlinear deterministic counterpart of the classical definitions given in [15] for linear time-invariant systems in a stochastic framework of quasi-stationary processes. These definitions clearly separate the concepts of identifiability, which is a property of the model structure, and of informativity, which is a property of the experimental data. In addition, we shall depart from the Linear Time Invariant (LTI) literature on identifiability and informativity by considering that the information content in the data, that allows estimation of the unknown parameters, can come either from the external input signal $u(\cdot)$ or from the response to an initial condition x_0 . Indeed in many engineering applications of nonlinear systems (e.g. in batch chemical processes) the experiment that allows the estimation of the unknown parameters is obtained by measuring the response of the nonlinear system to some initial condition; in particular, it is often the case that there are no external inputs to the system [19], [4], [10]. Example 1 in Section V illustrates this situation.

For the sake of generality and of possible future extensions, we present all definitions for the general model class (1), without any restriction on linearity in θ , rationality in x or polynomial structure in u .

Consider the model structure (1) at some value θ^* with initial condition x_0 :

$$\dot{x} = f(x, \theta^*) + g(x, u), \quad x(0) = x_0 \quad (7)$$

and the same model structure at θ with initial condition $\hat{x}_0$:

$$\dot{\hat{x}} = f(\hat{x}, \theta) + g(\hat{x}, u), \quad \hat{x}(0) = \hat{x}_0 \quad (8)$$

Definition 1: (Identifiability at θ^*) The model structure (1) is *globally identifiable at θ^** if there exists an experiment $z(\cdot) \triangleq \{u(\cdot), x_0\}$ such that, for all $\theta \in \mathbb{R}^d$, the solutions of the equations (7) and (8), driven by the same $u(\cdot)$ and with the same initial condition $x_0 = \hat{x}_0$ are identical (i.e. $\hat{x}(t, \theta) = x(t, \theta^*) \forall t \geq 0$) **only if** $\theta = \theta^*$.

This definition relies on the possible existence of an appropriate experiment $z(\cdot)$ that differentiates between different values of θ by measuring the evolution of the state. Such an experiment, when it exists, will be called informative. In [2], the new concept of *informativity at a particular θ^** was introduced for stationary stochastic LTI systems; the next definition is its deterministic counterpart.

Definition 2: (Informativity at θ^*) The experiment $z(\cdot) \triangleq \{u(\cdot), x_0\}$ is *globally informative at θ^** for the model set (1) if, for all $\theta \in \mathbb{R}^d$, the solutions of the equations (7) and (8), driven by the same experimental data $z(\cdot)$, are identical (i.e. $\hat{x}(t, \theta) = x(t, \theta^*) \forall t \geq 0$) **only if** $\theta = \theta^*$.

These definitions exhibit the two ingredients that are necessary for a meaningful identification: informativity (also called sufficient richness), which is a property of the applied experiment (input signal, initial condition, or a combination of these), and identifiability, which refers to the possible existence of an informative experiment given a particular

model structure. These two ingredients depend only on the model structure. We should add, however, that under our Assumption 3, the purpose of identification is to identify the true system. This requires that one is able to produce an informative experiment at θ_0 ; in that sense, an experiment that is informative for the identification of the true system depends on that system.

IV. FUNDAMENTAL IDENTIFIABILITY ANALYSIS

From now on, we analyze the identifiability of the model structure (5), under the Assumptions 1, 2 and 3. We first derive a property of this model structure that will be useful in our subsequent developments.

Lemma 4.1: At all $\theta \neq 0$ there exist x_0 and u for which $\dot{x}(0) \neq 0$.

Proof: (i) Consider first the case where at least one of the $g_i(x)$ is not the zero polynomial. Let then x_0 be any x for which at least one $g_i(x_0) \neq 0$, say $g_1(x_0) \neq 0$. Then clearly for all θ one can always chose u such that

$$g_1(x_0)u \neq -\theta^T \phi(x_0) - m(x_0) - \sum_{i=2}^l g_i(x_0)u^i$$

(ii) Now consider the case where there is no input, i.e. all $g_i(x)$ are zero. We show by contradiction that for all $\theta \neq 0$ there exists x_0 such that $\theta^T \phi(x_0) + m(x_0) \neq 0$. Indeed, suppose that there exists $\theta \neq 0$ such that $\theta^T \phi(x) + m(x) = 0 \forall x$, then this would violate Assumption 1 about $m(x)$. ■

We should comment that the rational nature of the model ensures that $\dot{x}(0) \neq 0$ for almost all x_0 and u . Rewriting the model structure (6) in the form

$$n(x)\dot{x} - m(x) - G(x)U = \phi^T(x)\theta, \quad x(0) = x_0 \quad (9)$$

shows that identifiability rests entirely on the vector $\phi(x)$. Since the left hand side is a measured quantity, we observe that θ can be uniquely determined from measured data over a period $[0, t]$ if and only if $\phi(x)$ can be made to span the whole space $\mathbb{R}^d$ over that same period.

Lemma 4.2: The model structure (9) is globally identifiable at some θ^* if and only if there exists an experiment $z(\cdot) \triangleq \{u(\tau) \text{ for } \tau \in [0, t], x_0\}$ and a scalar $\alpha > 0$ such that

$$\int_0^t \phi(x(\tau, \theta^*)) \phi^T(x(\tau, \theta^*)) d\tau > \alpha I \quad (10)$$

Proof: Denote by $x(\tau, \theta^*)$ the solution of (9) at θ^* . Premultiplying (9), evaluated at θ^* , by $\phi(\tau, \theta^*)$ and integrating from 0 to t yields

$$\begin{aligned} & \int_0^t \phi(\tau, \theta^*) [n(x(\tau, \theta^*)) \dot{x}(\tau, \theta^*) \\ & \quad - m(x(\tau, \theta^*)) - G(x(\tau, \theta^*))U] d\tau \\ & = \left[\int_0^t \phi(x(\tau, \theta^*)) \phi^T(x(\tau, \theta^*)) d\tau \right] \theta^* \end{aligned}$$

This equation has a unique solution in θ^* if and only if condition (10) holds. ■

The excitation condition (10) on the regressor vector $\phi(x(\tau, \theta)) \in \mathbb{R}^d$ hinges on two questions:

(i) does there exist a vector $X \triangleq [x_1, x_2, \dots, x_d] \in \mathbb{R}^d$ such that the $d \times d$ matrix $L(X) \triangleq [\phi(x_1), \phi(x_2), \dots, \phi(x_d)]$ is nonsingular?

(ii) does there exist an experiment that generates a $\phi(x(\tau, \theta))$ which spans the whole space $\mathbb{R}^d$? This requires that the trajectory $x(\tau, \theta)$, which depends on θ , $u(\cdot)$ and x_0 , passes through d points $\{x_1, x_2, \dots, x_d\}$ that make $L(X)$ nonsingular.

The first is a structural issue on $\phi(x)$ only; it is independent of θ , $u(\cdot)$, x_0 , $m(x)$, $n(x)$ and $G(x)$. Existence of such an X is a necessary condition for identifiability [6]. To proceed in this analysis we use the fact that $\phi(x)$ is an analytic function of time, which leads to the following result.

Theorem 4.1: The model structure (9) is globally identifiable at some θ^* if and only if there exists an experiment $z(\cdot) \triangleq \{u(\tau) \text{ for } \tau \in [0, t], x_0\}$ such that

$$\beta^T \phi(x_0) = \beta^T \dot{\phi}(x_0) = \dots = \beta^T \phi^{(k)}(x_0) = 0 \quad \forall k \geq 0 \quad (11)$$

implies $\beta = 0$.

Proof: It follows from Lemma 4.2 that the model structure (9) is **not** identifiable at θ^* if and only if for all initial conditions x_0 and for all inputs $u(\tau)$, $\tau \geq 0$,

$$\exists \beta \neq 0 : \quad \beta^T \phi(x(\theta^*, t)) \equiv 0 \quad \forall t \geq 0 \quad (12)$$

Because the solutions of (9) are analytic by Assumption 2, (12) is equivalent to the satisfaction of

$$\beta^T \phi(x) = \beta^T \dot{\phi}(x) = \dots = \beta^T \phi^{(k)}(x) = 0 \quad \forall k \geq 0 \quad (13)$$

for some $\beta \neq 0$ at any particular time, in particular at time $t = 0$, where $x(0) = x_0$. ■

The equivalence between (12) and (13) will form the basis for our derivations of identifiability and informativity in the following sections. To facilitate these derivations we now define the following $d \times (k+1)$ matrix

$$R_k(\theta, x) \triangleq [\phi(x) \quad \dot{\phi}(x) \quad \dots \quad \phi^{(k)}(x)], \quad k \geq 1 \quad (14)$$

In particular we shall consider

$$R_\infty(\theta, x) \triangleq [\phi(x) \quad \dot{\phi}(x) \quad \ddot{\phi}(x) \quad \dots] \quad (15)$$

Comment - It is important to realize that in the expressions (13)-(15) the derivatives of $\phi(x)$, evaluated at some time t_0 and hence at some state $x(t_0)$, depend also on θ and on the derivatives of x at t_0 . Equivalently, by application of the model equations, one can think of the derivatives of $\phi(x)$ as depending on θ , $u(t_0)$ and the derivatives of $u(t)$ at t_0 . In order not to be overwhelmed by notation, we shall only mention the θ -dependence of the matrices in these expressions.

With the notation (15), the statement (11) can be rewritten as

$$\beta^T R_\infty(\theta, x_0) = 0. \quad (16)$$

This leads to the following equivalent statement for Theorem 4.1.

Corollary 4.1: The model structure (9) is globally identifiable at some θ^* if and only if there exists an experiment $z \triangleq \{x_0, u(0), \dot{u}(0), \ddot{u}(0), \dots\}$ such that

$$\varrho(R_\infty(\theta^*, x_0)) = d \quad (17)$$

where $\varrho(\cdot)$ represents the rank of a matrix. ■

Now observe that

$$\begin{aligned} \dot{\phi}(x) &= \frac{\partial \phi(x)}{\partial x} \dot{x}, \\ \ddot{\phi}(x) &= \frac{\partial^2 \phi(x)}{\partial x^2} \dot{x}^2 + \frac{\partial \phi(x)}{\partial x} \ddot{x}, \end{aligned}$$

and so on for all time derivatives of $\phi(x)$. We define the following $d \times (k+1)$ matrix:

$$J_k(x) \triangleq \begin{pmatrix} \phi(x) & \frac{\partial \phi(x)}{\partial x} & \frac{\partial^2 \phi(x)}{\partial x^2} & \dots & \frac{\partial^k \phi(x)}{\partial x^k} \end{pmatrix} \quad (18)$$

which contains only the derivatives of $\phi(x)$ with respect to x and thus depends only on the structure of the vector $\phi(x)$ itself (and not on θ and $u(t)$). This matrix has the following property.

Lemma 4.3: For $k \geq q$ we have

$$J_k(x) = \begin{bmatrix} J_q(x) & 0_{d \times (k-q)} \end{bmatrix} \quad (19)$$

In addition the rank of $J_q(x)$ is the same for all x .

Proof: Expression (19) follows from $\deg(\phi(x)) = q$. Since $\phi(x)$ is a polynomial vector of degree q , it can always be written as $\phi(x) = T[1 \ x \ \dots x^{q-1} \ x^q]^T$ where $T \in \mathbb{R}^{d \times (q+1)}$. Since $\frac{\partial^i}{\partial x^i} \phi(x) = T \frac{d^i}{dx^i} [1, x, \dots, x^q]^T$, it follows from the definition of $J_q(x)$ that it can be written as

$$J_q(x) = TM(x), \quad (20)$$

where $M(x)$ is a $(q+1) \times (q+1)$ lower triangular matrix whose i -th diagonal element is $(i-1)!$. Hence $M(x)$ is nonsingular for all x , and $\varrho(J_q(x)) = \varrho(T)$. ■

We now show that $R_k(\theta, x)$ admits a factorization that will prove very useful in the sequel.

Lemma 4.4: For $k \geq q$ the matrix $R_k(\theta, x)$ defined in (14) can be factored as follows:

$$R_k(\theta, x) = J_q(x) \begin{bmatrix} W_q(\theta, \dot{x}, \ddot{x}, \dots, x^{(q)}) \\ \tilde{W}_{k-q}(\theta, \dot{x}, \ddot{x}, \dots, x^{(k)}) \end{bmatrix} \quad (21)$$

where W_q is a $(q+1) \times (q+1)$ upper-triangular matrix:

$$W_q(\theta, \dot{x}, \ddot{x}, \dots, x^{(q)}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \dot{x} & \ddot{x} & \dots & x^{(q)} \\ 0 & 0 & \dot{x}^2 & \dots & \times \\ \vdots & \vdots & \vdots & \ddots & \times \\ 0 & 0 & 0 & \dots & \dot{x}^q \end{pmatrix} \quad (22)$$

where all elements of $W_q(\theta, \dot{x}, \ddot{x}, \dots, x^{(q)})$ to the right of the diagonal contain derivatives of x .

Proof: Lengthy but straightforward calculations show that for $k \geq q$ one can always factorize $R_k(\theta, x)$ as $R_k(\theta, x) = J_k(x)W_k(\theta, \dot{x}, \ddot{x}, \dots, x^{(k)})$. It now follows from Lemma 4.3 that this factorization takes the form of (21)-(22), where $W_{k-q}(\theta, \dot{x}, \ddot{x}, \dots, x^{(k)})$ is the $(q+1) \times (k-q)$ top right block-matrix of $W_k(\theta, \dot{x}, \ddot{x}, \dots, x^{(k)})$. ■

The matrix $W_q(\theta, \dot{x}, \ddot{x}, \dots, x^{(q)})$ depends on the derivatives of the state and hence on θ . An easy way to compute these derivatives was presented in [9]. The results of Lemma 4.4 immediately lead to the following important *necessary and sufficient condition for identifiability*.

Theorem 4.2: The model structure (9) is globally identifiable at all $\theta \neq 0$ if and only if $\varrho(J_q) = d$. This requires in particular that $q \geq d-1$.

Proof: Since $R_\infty(\theta, x)$ can always be factored as in (21) and since (17) must hold at all θ , necessity is obvious. We prove sufficiency by contradiction. Thus, assume that $\varrho(J_q) = d$ but that the model structure is not identifiable at some $\theta^* \neq 0$. By Corollary 4.1 this means that for all x_0 and $u(\cdot)$ there exists $\beta(\theta^*, x_0) \neq 0$ such that

$$\beta(\theta^*, x_0)^T R_q(\theta^*, x_0) = 0. \quad (23)$$

Now $R_q(\theta^*, x_0) = J_q(x_0)W_q(\theta^*, \dot{x}(0), \ddot{x}(0), \dots, x^{(q)}(0))$ by (21). As $\theta^* \neq 0$, it follows from Lemma 4.1 that there exist a x_0 and a $u(\cdot)$ for which $\dot{x}(0) \neq 0$, which, from (22), implies that $W_q(\theta^*, \dot{x}(0), \ddot{x}(0), \dots, x^{(q)}(0))$ is nonsingular. Thus $\beta^T(\theta^*, x_0)J_q(x_0) = 0$ at this x_0 and $u(\cdot)$. From (20) and the fact that $M(x)$ is nonsingular for all x , it follows that $\beta(\theta^*, x_0)^T T = 0$ at this x_0 and $u(\cdot)$. As T is a constant matrix, there is thus a constant $\beta \neq 0$, such that $\beta^T T = 0$, which in turn implies:

$$\beta^T T M(x_0) = \beta^T J_q(x_0) = 0, \quad (24)$$

establishing the desired contradiction. ■

Theorem 4.2 gives a necessary and sufficient condition for identifiability, and also provides the tools to further advance in the analysis of identifiability and of informativity in different scenarios. The informativity issue has not been dealt with so far in this paper. On the other hand, identifiability as defined previously in this paper means that we can chose an input **and** an initial condition such that the identification will succeed. But it is seldom the case that the designer of an identification experiment has the choice of an input signal and at the same time can decide on the initial condition, so identifiability in this sense has limited practical interest.

Thus we shall consider from now on the following two situations separately, which are the ones of practical interest:

- 1) the system has no external input, i.e. $u(t) \equiv 0$, and identifiability must be secured through the initial condition $x(0)$. If the model structure is identifiable

through $x(0)$, then the informativity question amounts to finding a x_0 that delivers informative data via the transient response.

- 2) the input signal $u(\cdot)$ must provide informative data whatever the initial condition x_0 ; in this second situation, identifiability must be guaranteed by the input signal regardless of the initial condition, i.e. one must assure that “adversarial” initial conditions do not kill the excitation coming from $u(\cdot)$. If the model structure is identifiable through $u(\cdot)$, then the informativity question amounts to finding a sufficiently rich $u(\cdot)$ that provides informative data for all possible x_0 .

This second situation is the most commonly treated and probably the most commonly found in practice, but the first one is also found in a variety of applications, particularly in (bio-) chemical batch processes [19], [4], [10]. For pedagogical reasons, we start our analysis with the simpler case where the identifiability of the parameters must be secured through the response to an initial state, assuming that there is no driving input to the system.

V. IDENTIFIABILITY AND INFORMATIVITY FROM THE INITIAL STATE

A. Definitions

When $u(t) \equiv 0$ the models (7) and (8) become:

$$\dot{x} = f(x, \theta^*), \quad x(0) = x_0 \quad (25)$$

$$\dot{\hat{x}} = f(\hat{x}, \theta), \quad \hat{x}(0) = \hat{x}_0 \quad (26)$$

Definition 3: (Identifiability at θ^* from $x(0)$) The model structure (25) is *globally identifiable at θ^* from the initial condition $x(0)$* if there exists an initial condition $x_0 = \hat{x}_0$ such that, for all $\theta \in \mathbb{R}^d$, the solutions of the equations (25) and (26) are identical (i.e. $x(t) = \hat{x}(t) \quad \forall t \geq 0$) **only if** $\theta = \theta^*$.

Definition 4: (Informativity of the initial condition at θ^*) The initial condition x_0 is globally informative at θ^* for the model structure (25) if for all $\theta \in \mathbb{R}^d$, the solutions of the equations (25) and (26) with initial condition $x_0 = \hat{x}_0$ are identical (i.e. $x(t) = \hat{x}(t) \quad \forall t \geq 0$) **only if** $\theta = \theta^*$.

B. Identifiability and informativity results

We now provide necessary and sufficient conditions for identifiability and informativity at a given θ from the initial state, for the model class of rational models (5) in which the input is assumed zero:

$$\dot{x} = \frac{1}{n(x)}[\theta^T \phi(x) + m(x)], \quad x(0) = x_0 \quad (27)$$

The following theorem is our main result on identifiability from the initial state.

Theorem 5.1: The model class (27) is globally identifiable from the initial state at every $\theta \neq 0$ if and only if $\varrho(J_q) = d$.

Proof: By Corollary 4.1, the model class is globally identifiable at some θ^* if and only if there exists x_0 such that $\varrho(R_\infty(\theta^*, x_0)) = d$. By Lemma 4.4, $R_\infty(\theta^*, x_0)$ can be

factored as in (21), from which necessity follows immediately. We prove sufficiency by contradiction. Assume that $\varrho(J_q(x)) = d$ for all x , but that the model structure is not identifiable at some $\theta^* \neq 0$. By Corollary 4.1 this means that for all x_0 there exists $\beta(\theta^*, x_0) \neq 0$ such that

$$\beta(\theta^*, x_0)^T R_q(\theta^*, x_0) = 0. \quad (28)$$

By the same reasoning as in the sufficiency proof of Theorem 4.2, it then follows that for all x_0 we have $\dot{x}(0) = 0$ at $\theta = \theta^*$, or equivalently $\theta^{*T} \phi(x_0) + m(x_0) = 0$, using (27). But this would contradict Lemma 4.1. ■

We also have an immediate characterization of an informative initial condition.

Theorem 5.2: Let the model structure (27) be globally identifiable from the initial state at a given θ^* (i.e. $\varrho(J_q) = d$) and let $d > 1$. Then an initial condition $x_0 \in \mathbb{R}$ yields a globally informative experiment at θ^* if and only if x_0 is not a root of the polynomial equation

$$\phi^T(x_0)\theta^* + m(x_0) = 0. \quad (29)$$

Proof: Given that $n(x) > 0$ for all x , the result is a straightforward extension to the model class (27) of Theorem 5.3 of [11] developed for polynomial systems. ■

Note that the existence of such x_0 is guaranteed by Assumption 1. In fact, for any given θ almost all initial conditions are informative, since the polynomial $\theta^T \phi(x) + m(x)$ has a finite number of roots.

The condition $d > 1$ is included in Theorem 5.2 for the sake of making the statement necessary and sufficient. For $d = 1$, that is a scalar $\phi(x)$, θ may still be identifiable when $\phi(x_0)\theta + m(x_0) = 0$ provided that $\phi(x_0) \neq 0$.

Example 1

Consider a perfectly mixed batch reactor for which the following irreversible chemical reaction takes place:

That is, component A reacts irreversibly at a specific reaction rate r to form component B . Let x be the concentration of A . The reaction rate is often modeled as follows:

$$r(x, \theta_1, \theta_2) = \frac{\theta_1 x + \theta_2}{x + K} \quad (31)$$

with K denoting a given positive constant and θ_1, θ_2 being parameters to be identified.

Applying the continuity equation for the A component leads to the following dynamics

$$\dot{x} = -r(x, \theta_1, \theta_2)x = -\frac{1}{x + K}\theta^T \phi(x), \quad x \geq 0, \quad x(0) = x_0, \quad (32)$$

with $\theta^T = [\theta_1 \quad \theta_2]$ and $\phi^T(x) = [x^2 \quad x]$.

We first observe that $q = 2$ and

$$J_2(x) = \begin{pmatrix} x^2 & 2x & 2 \\ x & 1 & 0 \end{pmatrix} = TM(x)$$

with

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M(x) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & 2x & 2 \end{pmatrix}$$

so that $\rho(J_2(x)) = 2$ for all x ; hence the model structure is globally identifiable from the initial state at every θ . The factorization (21) with $k = q = 2$ yields the 2×3 matrix $R_2(\theta, x)$ as:

$$R_2(\theta, x) = J_2(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dot{x} & \ddot{x} \\ 0 & 0 & \dot{x}^2 \end{pmatrix} = J_2(x)W(\theta, \dot{x}, \ddot{x}) \quad (33)$$

Theorem 5.2 states that an initial state x_0 is informative at some θ if and only if $\theta_1 x_0^2 + \theta_2 x_0 \neq 0$. But $\theta_1 x_0^2 + \theta_2 x_0 = 0$ implies either $x_0 = 0$ or $x_0 = -\frac{\theta_2}{\theta_1}$. Any initial condition other than these two yields $\theta_1 x_0^2 + \theta_2 x_0 \neq 0$.

Therefore any initial condition other than $x_0 = 0$ or $x_0 = -\frac{\theta_2}{\theta_1}$ is informative. One of these uninformative initial conditions is a function of the unknown θ . Thus we conclude that two experiments with two different nonzero initial conditions will always allow the estimation of the parameter vector θ .

VI. IDENTIFIABILITY AND INFORMATIVITY FROM THE INPUT

We now seek identifiability conditions and informativity from the input *regardless of the initial state*. The definitions of Section III can be adapted to this case as follows.

A. Definitions: Identifiability and informativity from $u(\cdot)$

Consider the model structure

$$\dot{x} = \frac{\theta^* T \phi(x) + m(x) + G(x)U}{n(x)}, \quad x(0) = x_0 \quad (34)$$

and the same model structure at some other parameter vector θ with the same initial condition:

$$\dot{\hat{x}} = \frac{\theta^T \phi(\hat{x}) + m(\hat{x}) + G(\hat{x})U}{n(\hat{x})}, \quad \hat{x}(0) = x_0 \quad (35)$$

Definition 5: (Identifiability at θ^* from u) The model structure (34) is *globally identifiable at θ^* from the input u* if there exists an input $u(\cdot)$ such that, for all initial conditions x_0 and for all $\theta \in \mathbb{R}^d$, the solutions of the models (34) and (35) are identical (i.e. $x(t) = \hat{x}(t) \quad \forall t \geq 0$) **only if** $\theta = \theta^*$.

This definition is consistent with the definition of global identifiability at some θ^* used in [15] and adopted in [2] for LTI systems.

Definition 6: (Informativity of the input at θ^*) The input signal $u(\cdot)$ is globally informative at θ^* for the model structure (34) if, for all initial conditions x_0 and for all $\theta \in \mathbb{R}^d$, the solutions of the models (34) and (35) with this input $u(\cdot)$ are identical (i.e. $x(t) = \hat{x}(t) \quad \forall t \geq 0$) **only if** $\theta = \theta^*$.

B. Identifiability from the input

The following theorem is the main result on identifiability from the input.

Theorem 6.1: Consider the model structure (34) with $\deg(\phi(x)) = q$ and let $d > 1$. This model structure is globally identifiable at θ^* from the input u if and only if the following two conditions hold simultaneously:

- (i) $\rho(J_q) = d$;
- (ii) the polynomials $\theta^{*T} \phi(x) + m(x)$ and $\{g_i(x), i = 1, \dots, l\}$ have no common real root w.r.t. x .

Proof: First note that, from Definition 5 and Theorem 4.1, identifiability at θ^* from u is equivalent to the existence of an input $u(\cdot)$ such that $\rho(R_\infty(\theta^*, x_0)) = d \quad \forall x_0$. Since this condition is identical to that of Theorem 1 of [6] the result follows from that Theorem. ■

The condition $d > 1$ is there for the sake of making the statement necessary and sufficient. For $d = 1$ we have the following

Corollary 6.1: Consider the model structure (5) with $\deg(\phi(x)) = q$ and let $d = 1$. This model structure is identifiable at θ^* from the input u if the following two conditions hold simultaneously:

- (i) $\rho(J_q) = 1$;
- (ii) the polynomials $\theta^* \phi(x) + m(x)$ and $\{g_i(x), i = 1, \dots, l\}$ have no common real root w.r.t. x .

Moreover, condition (ii) is not necessary for identifiability from the input.

Proof: From Theorem 6.1, under the necessary condition $\rho(J_q) = d$, identifiability from the input is equivalent to $\theta^{*T} \phi(x_0) + m(x_0) + G(x_0)U \neq 0$ for all $x_0 \in \mathbb{R}$. This, in its turn, is implied by condition (ii), but only implies condition (ii) for $d > 1$. Indeed, for $d = 1$, if $\theta^* \phi(x_0) + m(x_0) = 0$ and $\{g_i(x_0) = 0, i = 1, \dots, l\}$ for some x_0 , then θ^* may still be identified from the equation $\phi(x_0)\theta^* + m(x_0) = 0$ provided that $\phi(x_0) \neq 0$. ■

C. Informativity from the input

If the identifiability conditions of Theorem 6.1 are fulfilled, then there exists an informative input signal, that is, one that allows the discrimination between two different parameter vectors θ and θ^* regardless of the initial condition. However, Theorem 6.1 does not provide a clue as to which input signals will be informative. Obtaining necessary and sufficient conditions on the input signal u to be informative for the model class (34) regardless of the initial condition turns out to be difficult. The objective of the present subsection is twofold: we first provide a sufficient condition for informativity of the signal u ; we then illustrate with an example that this condition is not necessary, i.e. for this example we construct an informative input that does not satisfy this sufficient condition.

Theorem 6.2: Assume that the model structure (34) is globally identifiable at some θ^* , i.e. the conditions of Theorem 6.1 hold. An input signal $u(\cdot)$ is informative at θ^* if at any given time, say $t = 0$, $\dot{x} \neq 0 \quad \forall x_0$.

Proof: It follows from Definition 6 that an input signal $\{u(\tau), \tau \geq 0\}$ is informative at θ^* if and only if the matrix $R_\infty(\theta^*, x_0)$ has full row rank for all $x_0 \in \mathbb{R}$. Now consider $R_q(\theta^*, x_0)$ and factorize this matrix as in (21):

$$R_q(\theta^*, x_0) = J_q(x_0)W_q(\theta^*, \dot{x}, \dots, x^{(q)}) \quad (36)$$

where the derivatives in W_q are evaluated at $t = 0$. It follows from (22) that W_q is nonsingular for all x_0 if $\dot{x}(0) \neq 0$ for all x_0 . Since $\varrho(J_q) = d$ by the global identifiability assumption, it then follows from the Sylvester inequality applied to (36):

$$\varrho(J_q) + \varrho(W_q) - (q + 1) \leq \varrho(R_q),$$

that $\varrho(R_q(\theta^*, x_0)) = d$ for all x_0 if $\dot{x}(0) \neq 0$ for all x_0 . Since $R_q(\theta^*, x_0)$ is a submatrix of $R_\infty(\theta^*, x_0)$, the result follows. ■

The sufficient condition on u can equivalently be stated as follows.

Corollary 6.2: Assume that the model structure (34) is globally identifiable at some θ^* . An input signal $u(\cdot)$ is informative at θ^* if

$$\theta^{*T} \phi(x_0) + m(x_0) + \sum_{i=1}^l g_i(x_0)u^i \neq 0 \quad \forall x_0. \quad (37)$$

The following example shows that the condition of Theorem 6.2 is not necessary; but it also illustrates that there may be no input that satisfies the condition (37) even though an informative input exists. We shall construct such informative input by directly ensuring that the matrix $R_q(\theta^*, x_0)$ has full rank for all x_0 .

Example 2

Consider a continuous stirred-tank reactor for which the irreversible reaction in (30) takes place, with r being as in (31). From the continuity equation for the A component, the following model structure is obtained (see e.g. [17]):

$$\begin{aligned} \dot{x} &= -r(x, \theta_1, \theta_2)x + (C - x)u = \\ &= -\frac{1}{x + K}[\theta^T \phi(x) + (x - C)(x + K)u], \\ &x \geq 0, \quad x(0) = x_0, \quad C > x_0, \end{aligned} \quad (38)$$

with x being the concentration of A , u the dilution rate, C the inlet concentration of A , K a given positive constant, $\theta^T = [\theta_1 \quad \theta_2]$ and $\phi^T(x) = [x^2 \quad x]$.

Consider first the identifiability conditions of Theorem 6.1. We note that $q = 2$ and that $J_2(x)$ is as in Example 1, so that $\varrho(J_2(x)) = 2$ for all x . Defining $g(x) \triangleq (x - C)(x + K)$, and noting that $C > 0$, $K > 0$ and $x \geq 0$, we observe that the polynomials $\theta^T \phi(x)$ and $g(x)$ have a common root only if $\theta_2 = -C\theta_1$. The model structure is therefore identifiable at all θ except those for which $\theta_2 = -C\theta_1$.

Now consider informativity at any identifiable θ . We observe that there is no input $u(\cdot)$ that makes $\dot{x}(0) \neq 0$ for all x_0 ; thus the sufficient condition of Theorem 6.2 cannot be used to construct an informative input. However, we show

that one can construct an input $u(\cdot)$ that makes $R_2(\theta, x_0)$ full rank for all x_0 at all identifiable θ . Notice that $R_2(\theta, x)$ takes the form:

$$R_2(\theta, x_0) = \begin{pmatrix} x_0^2 & 2x_0\dot{x}(0) & 2x_0\ddot{x}(0) + 2\dot{x}^2(0) \\ x_0 & \dot{x}(0) & \ddot{x}(0) \end{pmatrix}$$

We observe that the determinants of the three minors of R_2 are, respectively, $-x_0^2\dot{x}(0)$, $-2\dot{x}^2(0)$ and $(-x_0^2\ddot{x}(0) - 2x_0\dot{x}^2(0))$. Substitute the following equations evaluated at 0

$$\begin{aligned} \dot{x} &= -\frac{1}{x + K}[\theta_1 x^2 + \theta_2 x + (x - C)(x + K)u] \\ \ddot{x} &= -\frac{1}{x + K}[2\theta_1 \dot{x}x + \theta_2 \dot{x} + (x - C)(x + K)\dot{u} \\ &\quad + 2x\dot{x}u + (K - C)\dot{x}u + \dot{x}^2] \end{aligned}$$

and consider 2 cases separately: $x_0 = 0$ and $x_0 \neq 0$.

Case 1: Suppose first that $x_0 = 0$; then $\dot{x}(0) = Cu(0)$, $\ddot{x}(0) = \sigma_0 \triangleq -\frac{1}{K}[\theta_2 Cu(0) - KC(u^2(0) - \dot{u}(0))]$, and

$$R_2(\theta, 0) = \begin{pmatrix} 0 & 0 & 2C^2u^2(0) \\ 0 & Cu(0) & \sigma_0 \end{pmatrix}$$

Thus, $R_2(\theta, 0)$ has full rank if and only if $u(0) \neq 0$.

Case 2: Take $u(0) \neq 0$ and consider $x_0 \neq 0$. Again two cases must be considered.

- (a) Suppose first that $\dot{x}(0) \neq 0$. Then $R_2(\theta, x_0)$ has full rank because the determinant of the second minor $-2\dot{x}^2(0) \neq 0$.
- (b) Suppose next that $\dot{x}(0) = 0$. Then the determinant of the third minor of $R_2(\theta, x_0)$ is given by $-x_0^2\ddot{x}(0) = x_0^2(x_0 - C)\dot{u}(0)$. Given the constraint $C > x_0$ and the assumption $x_0 \neq 0$, it follows that $-x_0^2\ddot{x}(0) \neq 0$ if one takes $\dot{u}(0) \neq 0$.

We conclude that, at any identifiable θ , an input $u(\cdot)$ such that at some time, say $t = 0$, $u(0) \neq 0$ and $\dot{u}(0) \neq 0$ is globally informative for the model structure (38). These informativity conditions are independent of θ .

Example 2 illustrates an interesting feature of the theory developed in this paper. For this example, we have $q + 1 > d$. It then follows from the factorization of R_q in (33) that $\dot{x}(0) \neq 0$ is not a necessary condition for the full rankness of R_q . Indeed we observe in our example that, even if $\dot{x}(0) = 0$, we can still find an informative input by taking $\dot{u}(0) \neq 0$. Conversely, there is no input that makes $\dot{x}(0) \neq 0$ for all x_0 .

VII. CONCLUSIONS

In this paper we have provided necessary and sufficient conditions for identifiability from the initial state, as well as necessary and sufficient conditions for informativity from the initial state for a class of nonlinear rational models. Parameter estimation from a transient response is of great interest in many practical applications. In addition, for this same class of models we have provided necessary and sufficient conditions for identifiability from the input signal regardless of the unknown initial condition. Sufficient conditions for the generation of informative experiments from the input have also been derived, and we have illustrated with an

example the difficulty of obtaining necessary and sufficient informativity conditions for such case.

The properties that make an initial state and/or an input globally informative at given θ are generic. But informativity, as well as identifiability, refers to a given model structure at a given parameter value; so for any given experiment there are parameter values for which it is not informative. Since the properties of identifiability and informativity should be valid at the unknown real parameter, care must still be exercised in avoiding noninformative experimental conditions.

REFERENCES

- [1] E. W. Bai and S. S. Sastry. Persistence of excitation, sufficient richness and parameter convergence in discrete time adaptive control. *Systems and Control Letters*, 6:153–163, 1985.
- [2] A.S. Bazanella, X. Bombois, and M. Gevers. Necessary and sufficient conditions for uniqueness of the minimum in prediction error identification. *Automatica*, 48(8):1621–1630, August 2012.
- [3] A.S. Bazanella, M. Gevers, D. Coutinho, and R. Rui. Identifiability and excitation of a class of rational systems. In *USB Proc. of 53rd IEEE Conference on Decision and Control*, pages 6522–6527, Los Angeles, December 2014.
- [4] Dochain D. State and parameter estimation in chemical and biochemical processes: a tutorial. *Journal of Process Control*, 13:801–818, 2003.
- [5] A.G. Dankers, R. Toth, P.S.C. Heuberger, X. Bombois, and P.M.J. Van den Hof. Informative data and identifiability in lpv-arx prediction-error identification. In *CD-ROM of 50th IEEE Conf. on Decision and Control and European Control Conference*, pages 799–804, Orlando, Florida, USA, December 2011. IEEE.
- [6] S. Dasgupta, M. Gevers, G. Bastin, G. Campion, and Libei Chen. Identifiability of scalar linearly parametrized polynomial systems. In *Proc. 9th IFAC Symposium on Identification and System Parameter Estimation (SYSID 91)*, volume 1, pages 374–378, Budapest, Hungary, July 1991.
- [7] S. Dasgupta, Y. Shrivastava, and G. Krenzer. Persistent excitation in bilinear systems. *IEEE Transactions on Automatic Control*, 36(3):305–313, March 1991.
- [8] S. Diop and M. Fliess. Nonlinear observability, identifiability and persistent trajectories. In *Proc. 30th IEEE Conf. on Decision and Control*, pages 714–719, Brighton, UK, December 1991.
- [9] M. Fliess, C. Join, and H. Sira-Ramirez. Non-linear estimation is easy. *Int. J. Modelling, Identification and Control*, 4(1):12–27, 2008.
- [10] Franceschini G. and Macchietto S. Model-based design of experiments for parameter precision: State of the art. *Chemical Engineering Science*, 63:4846–4872, 2008.
- [11] M. Gevers, A. Bazanella, D. Coutinho, and S. Dasgupta. Identifiability and excitation of polynomial systems. In *CD-ROM of 52nd IEEE Conf. on Decision and Control*, pages 4278–4283, Florence, Italy, December 2013.
- [12] M. Gevers, A.S. Bazanella, X. Bombois, and L. Mišković. Identification and the information matrix: how to get just sufficiently rich? *IEEE Transactions on Automatic Control*, 54(12):2828–2840, December 2009.
- [13] G. Kreisselmeier. Richness and excitation on an interval with application to continuous-time adaptive control. *IEEE Transactions on Automatic Control*, 35(2):165–171, February 1990.
- [14] J.S. Lin and I. Kanellakopoulos. Nonlinearities enhance parameter convergence in output-feedback systems. *IEEE Transactions on Automatic Control*, 43(2):204–222, February 1998.
- [15] L. Ljung. *System Identification: Theory for the User, 2nd Edition*. Prentice-Hall, Englewood Cliffs, NJ, 1999.
- [16] L. Ljung and T. Glad. On global identifiability for arbitrary model parametrizations. *Automatica*, 30(2):265–276, 1994.
- [17] W.L. Luyben. *Process modeling, simulation, and control for chemical engineers*. McGraw-Hill chemical engineering series. McGraw-Hill, 1990.
- [18] I.M.Y. Mareels and M. Gevers. Persistence of excitation criteria for linear, multivariable, time-varying systems. *Mathematics of Control, Signals and Systems*, 1(3):203–226, 1988.
- [19] A. Vrsalovic Presecki and D. Vasic-Racki. Modelling of the alcohol dehydrogenase production in baker’s yeast. *Process Biochemistry*, 40:2781–2791, 2005.
- [20] E.D. Sontag, Y. Wang, and A. Megretski. Input classes for identifiability of bilinear systems. *IEEE Transactions on Automatic Control*, 54(2):195–207, February 2009.