Escape time formulation of state estimation and stabilization with quantized intermittent communication

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Abstract

The problems of state estimation and feedback stabilization of a linear system including a communications channel are quantified as escape or survival times, which yield stochastic processes describing the time of first exit of the state estimate error or of the system output from a specific domain. The complications introduced by communications – intermittency, channel noise, quantization, etc – are evaluated using a Markov stopping time formulation. This is compared to and contrasted with earlier analyses which considered the behavior of Kalman filters with intermittent data based on moments and conditional moments, and the evaluation of the minimal number of bits required for mean square stabilization. The main result shows the escape time is characterized by a Markov chain which is amenable to explicit analysis through the calculation of its cumulative distribution function. This is examined in the linear Gaussian and quantized linear Gaussian cases and then used to develop an approach to bitrate assignment in such communications-based control systems.

1 Introduction

We study the problems of state estimation and output feedback stabilization of a time-invariant linear system including a single communications link.

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad (1)$$

$$y_k = Cx_k + v_k, \quad (2)$$

$$z_k = \gamma_k Q_d(y_k). \quad (3)$$

Here, as usual, x_k , u_k , y_k , w_k , v_k are the system state, input, output, process noise and measurement noise signals of dimensions n , p , m , n , m respectively and $[A, B, C]$ are the system matrices of conformable dimensions. The signal z_k is the received measurement from the system and is modeled in (3) as the product of a scalar random variable, γ_k , taking values 0 or 1 and a quantized version of the system output y_k . As in [20, 19], the intermittency of the communication channel is modeled by the $\{\gamma_k\}$ sequence. The quantization function $Q_d(\cdot)$ is a subtractive dithered quantizer with finite bitrate in each channel of input y_k and will be further explicated shortly.

Definition 1 (Escape time) Given a closed domain $\mathcal{D} \subset$

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$\mathbb{R}^d$ and a stochastic process $\{\xi_k : k = 1, \dots\}$ on $\mathbb{R}^d$, the escape time is defined to be

$$\tau_e = \begin{cases} \arg \min_k \xi_k \notin \mathcal{D}, & \text{or} \\ \infty, & \text{if } \xi_k \in \mathcal{D} \quad \forall k. \end{cases}$$

Sometimes the escape time is called the ‘first exit time,’ ‘stopping time,’ ‘hitting time’ or ‘residence time.’ We shall be concerned with the escape time for the state process, x_k , state prediction error process, $\tilde{x}_{k|k-1}$, or the output process, y_k , of (1-2) when the control input is causally computed.

Figure 1 shows two simulations of the Kalman state prediction error $\tilde{x}_{k|k-1}$ for a scalar system with a magnitude bound placed at 60 and differing rates, P_γ , of packet arrival. In the left graph, the expected value of the conditional error covariance – the focus of [20] – is finite, while in the right it is not. Note the similarities between the figures except for the time scales. Escape time would correspond to the first achievement of the bound 60. Our interest is in characterizing this escape time cumulative distribution function.

Clearly the escape time is a random variable provided the infinite value has zero probability. We have the following result applicable in the linear Gaussian case and independent of the system matrices $[A, B, C]$.

Lemma 1 For the linear system (1), with noise process $\{w_k\}$ Gaussian, white, possessing full-rank covariance and

Fig. 1. Simulation of Kalman predictor error with $A=1.2$, $C = 1$, $Q = 0.005$, $R = 0.001$, $P_\gamma = 0.15$ (left) and 0.1 (right), and bound level 60.

independent from x_1 , with control u_k causally computed, and for $\mathcal{D}$ compact, the escape time of x_k is almost surely finite.

The proof is in the Appendix. The import of Lemma 1 is that it ensures that, in the linear Gaussian case or equivalent problems able to be transformed to linear Gaussian, using say Girsanov's Theorem, the finite escape of the state and/or output from any compact domain is ensured. The analysis of such processes then ought to concentrate on the description of the escape time rather than attempting to establish almost sure confinement to a compact set or characterize moment properties. This hearkens back to the escape time or residence time analysis of, say, [10, 24, 32, 12, 5]. These earlier treatments focus on stable continuous-time systems with small stochastic perturbations and use the Theory of Large Deviations to develop escape time characterizations as the noise power tends to zero. Our approach will maintain discrete time and deal with both stable and unstable systems with non-infinitesimal perturbations. This will not draw on Large Deviations Theory other than for comparison.

Our treatment of (1-3) endeavors to blend two distinct trains of research. The first is associated with the behavior of state estimators for such systems as treated in, say, [20, 19, 9, 16] with or without control being applied. Since the system is linear and if the applied control is known, the controlled state behavior is derivable from the estimator. The second class of problems, characterized by results such as [23, 22, 17, 31] concentrates on the stabilization aspects of the feedback control. The distinction between the two sets of problems in the literature rests with the description of the communications channel and the adaptation of quantization. The work in [15, 14] studies the earlier work in a more general case and yields necessary conditions for stabilization, recovering results of some previous works in the two approaches. In the estimator problem, the communication is taken to be intermittent — that is, the stochastic process $\{\gamma_k\}$ operates in a persistent fashion to cause arbitrarily long outages of

communications — but the communication is not limited in bitrate (there is no quantizer) and full state reconstruction occurs with any successful communication packet. In earlier work on the stabilization problem, the emphasis is on the quantizer and its associated bitrate limit and the channel is assumed not intermittent, i.e. $\gamma_k = 1$ for all k , with a deterministic maximal delay and possible additive channel noise. The approach adopted in this paper is to permit both intermittency and limited bitrate, since the Markov model describing escape time applies to both. We also pose a different set of questions dealing with escape time, which we regard as being more apropos for these problems. These focus not on limiting behaviors or mean-square stabilization but on characterizing the cumulative probability distribution function (cdf) of the escape time of the system state, output or state estimate error, since in general there is no almost sure bound on these, as stated in Lemma 1.

Before launching into the analysis, it is pertinent to examine some practical sources of estimation and control problems associated with systems described by (1-3), since the presence of a single communications link rules out teleoperation-styled feedback control problems. Utility management of a geographically distributed system, such as a power grid or radar network, where the sensors, but not the actuators, are remotely placed and linked back to base by communications networks, is the clearest application of state estimation operating with communications limits. The study of sensor fusion and its sibling area of sensor scheduling [3, 4] has a long history in these arenas. Schweppe [18] was a pioneer in the application of such methods in power system state estimation using data of variable reliability. More generally, the study of missing data has been longstanding in statistics [11] and in array beamforming [21] with studies of estimation in high noise going back to Wiener [26], whose work was connected with the origins of radar.

Closing the loop on a system to achieve stabilization using communicated data would appear to be a more recent problem. Interestingly, Wong and Brockett study first the state estimation problem [29] and then the feedback stabilization problem [30] for systems with reliable but bandlimited communications. This extends earlier results due to Williamson [27, 28] and Delchamps [2] on finite-wordlength effects on estimation and control in deterministic contexts. Sensor scheduling is also a feedback control or decision problem with a solution achievable via dynamic programming. The unifying aspect of these earlier analyses of the estimation and the control problems is that the covariance function of the state estimate error or of the state itself is the target of the analysis. Thus, mean-square stabilization is the objective in Nair and Evans [17].

For communications problems with intermittence, the conditional covariance, $\Sigma_{k+1|k}$, of the state prediction error is a random process adapted to the $\{\gamma_k\}$ sequence. Recent works have been concerned with the distributional and moment properties of this $\{\Sigma_{k+1|k}\}$ process. Thus: Sinopoli et al. [20] consider the convergence of the expectation $E[\Sigma_{k+1|k}]$

as $k \rightarrow \infty$; [8] models the packet dropping channel as the two state Markov chain and arrives at the stabilization condition as in [20] for the scalar case; Shi, Epstein and Murray [19] quantify the probability $\Pr(\Sigma_{k+1|k} > G)$ for given matrix G ; [9] analyzes conditions for the weak (in distribution) convergence of $\Sigma_{k+1|k}$; and, [16] treats the *tail* distribution properties of this covariance. (In this latter reference ‘tail’ refers to the distribution on the tail σ -algebra as $k \rightarrow \infty$ and not the tail probabilities in the sense of $\Sigma_{k+1|k}$ taking on large values for any specific value of k .)

The problem formulation is presented in Section 2 and the concepts of escape and survival times are introduced for the estimation and stabilization problems. Section 3 develops the central Markov chain description of the escape time problem in a general context before examining this for Gaussian and quantized Gaussian linear systems. These results pertain when the packet arrival sequence, $\{\gamma_k\}$, is known. Section 4 presents numerical examples to demonstrate: the method, heuristics of escape time, and the quality of approximations. The analysis is then extended to explore approaches to quantization and retransmission in the output stabilization escape time. The Appendix follows the conclusion and contains the proofs.

Some of the material of this paper was presented in preliminary form in [7]. This present paper completes these results for the estimation problem by including theoretical, quantization and bit-rate assignment analyses together with computational approaches to bounding the escape time cdfs. This paper also extends the results of [7] to encompass output feedback stabilization of the underlying control system. Lastly, this paper corrects an embarrassing error in Theorem 3 dealing with the computation of the escape cdf. This error was pointed out by the reviewers of this paper, for which we are very grateful and for which we provide some comment.

2 Problem formulation

2.1 Assumptions

We commence with the communications-linked control system described by (1-3) and make the following assumptions.

Assumption 1 1.1 x_1 , the initial state, is Gaussian with mean $\bar{x}_1$, and covariance $\Sigma_{1|0}$.

1.2 $u_k = -K\hat{x}_{k|k}$ constant linear state filter feedback is applied.

1.3 w_k and v_k are independent, white, zero-mean Gaussian processes independent from x_1 and with covariance matrices Q and R , which we take to be positive definite.

1.4 $\{\gamma_k\}$ is a Bernoulli random process independent from x_1 , $\{w_k\}$, and $\{v_k\}$ and taking values 0 or 1 to describe the non-arrival or arrival of a data packet, respectively.

1.5 $\mathcal{Q}_d$ is a subtractive, dithered, b -bit-per-channel, mid-rise, symmetric, linear quantizer with saturation values $\pm\zeta$. The subtractive dither signal is white, triangularly distributed, $\text{tr}(-\zeta/2^{b-1}, \zeta/2^{b-1})$, independent from all other signals on the transmitter side, and known exactly to both transmitter and receiver.

A number of extensions are possible at the expense of clarity of development. (i) Time-variation can be included into the noise covariance matrices and into the feedback control gain. Stationarity of the problem does permit the calculation of expectations which converge over time as in [20]. However given Lemma 1, our focus is not on limiting behaviors but explicitly on transient properties, where time-variation is readily accommodated. (ii) In Assumption 1.4 we assume the process $\{\gamma_k\}$ is Bernoulli, as in [20]. It might equally well be taken as Markov as in [9]. (iii) Extension to non-linear forms of quantization is straightforward. Extension to adaptive quantizers, while it is at the heart of Nair’s and Evans’ [17] demonstration of the minimal feedback bitrate required for mean-square stabilization, comes at the cost of communications and is not part of our analysis.

2.2 Incorporating quantization

Subtractive dithered b -bit quantizer, $\mathcal{Q}_d(\cdot)$, is depicted in Figure 2. It includes the fixed quantizer $\mathcal{Q}$ and subtractive dither signal, d_k .

Fig. 2. Subtractive dithered quantization with intermittent gain γ_k .

Fig. 3. The linear quantizer $\mathcal{Q}$

The fixed quantizer function $\mathcal{Q}(\cdot)$ is a mid-rise, symmetric, linear quantizer with 2^b levels within the range $[-\zeta, \zeta]$.

Figure 3 shows a 3-bit quantizer $\mathcal{Q}$, the basic unit of the quantizer is $\Delta = \zeta/2^{b-1}$. We note that subtractive dithered quantization requires careful handling of the wordlength of the subtraction operation, since d_k is recorded at a higher bitrate than b bits, see [25].

Theorem 1 *Under the conditions of Assumption 1.5, and provided $y_k \in [-\zeta, \zeta]$, the quantization noise,*

$$n_k = \mathcal{Q}_d[y_k] - y_k = [\mathcal{Q}(y_k + d_k) - d_k] - y_k,$$

is white, independent from $\{y_k\}$, and uniformly distributed $U(-\zeta/2^b, \zeta/2^b)$.

Quantization noise n_k is a deterministic function of y_k and d_k . The independence of n_k and y_k and the uniform distribution of n_k from the theorem statement follow directly from Theorem QTSD of [25]. The whiteness of $\{n_k\}$ itself and its independence from $\{y_j : j \neq k\}$ follow from the whiteness of $\{d_k\}$ and its independence from $\{y_k\}$.

The effective measurement noise in z_k is the sum of two independent terms, one Gaussian (measurement noise) and the other uniform (quantization noise).

$$v_{\text{eff},k} = v_k + n_k, \quad (4)$$

In our subsequent analysis, we show how to compute the escape times with any measurement noise process but for clarity specialize to either a strictly Gaussian measurement noise or a Gaussian-plus-uniform measurement noise. With independent m -vectors $v_k \sim N(0_m, R)$ and $n_k \sim U^m(-\zeta/2^b, \zeta/2^b)$ the pdf of $v_k + n_k$ is given by

$$f_{v+n}(z) = \frac{2^{b-1}}{\zeta} \left\{ \Phi \left[R^{-1/2} \left(z + \frac{\zeta}{2^b} \right) \right] - \Phi \left[R^{-1/2} \left(z - \frac{\zeta}{2^b} \right) \right] \right\}, \quad (5)$$

where $\Phi(\cdot)$ is the multivariate standard normal cdf of appropriate dimension. This is the convolution of the Gaussian pdf of v_k with the uniform pdf of n_k .

When a quantizer is present and for $y_k \in [-\zeta, \zeta]$, the effective measurement noise covariance for z_k increases from R with perfect reconstruction of y_k to

$$R_{\text{eff}} = R + \frac{\zeta^2}{3 \times 2^{2b}} I_m. \quad (6)$$

which is the measurement noise, v_k , covariance plus the quantizer noise, n_k , covariance since these two noises are additive in (4) and are independent. With quantization, this quantity replaces R in the Kalman filter recursion, which relies solely on second-order statistics and yields the least mean squares linear unbiased state estimator. (See Section 3.2 of [1].)

Outside of the range $[-\zeta, \zeta]$ the quantization error becomes potentially unbounded and correlated. In our situation of escape time analysis with fixed quantization, once the outer levels are breached the state or state estimation error has escaped. Nair and Evans in [17] develop an ingenious approach to adaptive quantization amenable to mitigating the effects of saturation errors by expanding ζ faster than the state can escape and, further, to achieving this within the allocated bitrate. In Section 5, strategies are studied to mitigate the effects of intermittency through the allocation of a fraction of the b bits in the communicated signal to the coarse retransmission of earlier values of y_k . These studies extend earlier studies of signal reconstruction with missing data [6] and can deal with strategies for bit assignment in both TCP/IP and UMDP settings depending on the transmitter's knowledge of the arrival of packets.

2.3 Kalman state estimation with intermittent quantized observations

The Kalman state estimation equations with intermittent quantized observations described by $\{\gamma_k\}$ and $\mathcal{Q}$, commencing from initial values $\hat{x}_{1|0} = \bar{x}_1$ and $\Sigma_{1|0}$, are as follows. These equations are identical to those of [20, 19] and subsequent works modulo the incorporation of quantization via R_{eff} and n_k .

measurement update

$$L_k = \gamma_k [\Sigma_{k|k-1} C^T (C \Sigma_{k|k-1} C^T + R_{\text{eff}})^{-1}], \quad (7)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (z_k - C \hat{x}_{k|k-1}),$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - L_k C \Sigma_{k|k-1}. \quad (8)$$

time update

$$\hat{x}_{k+1|k} = A \hat{x}_{k|k} + B u_k,$$

$$\Sigma_{k+1|k} = A \Sigma_{k|k} A^T + Q, \quad (9)$$

filtered error

$$\tilde{x}_{k|k} = x_k - \hat{x}_{k|k},$$

prediction error

$$\begin{aligned} \tilde{x}_{k+1|k} &= x_{k+1} - \hat{x}_{k+1|k}, \\ &= (A - A L_k C) \tilde{x}_{k|k-1} + w_k - A L_k (v_k + n_k). \end{aligned} \quad (10)$$

Note the γ_k appears explicitly solely in the Kalman gain L_k and that, conditioned on $\{\gamma_k\}$, the state prediction error system (10) is linear, time-varying due to γ_k , and driven by not-necessarily-Gaussian white noise. The Kalman predictor error equation (10) is central to both the estimator and the output escape time formulations below. We note that, when $\gamma_k = 0$, then $L_k = 0$ and (10) will be unstable if A has eigenvalues outside the unit circle and driven by w_k alone. If $\gamma_k = 0$ for all k — a case where we refer to the escape time as the *survival time* — (10) is time-invariant, which simplifies the analysis.

2.4 Output feedback control with intermittent and quantized observations

The output feedback control problem with intermittent and quantized observations differs from the estimator problem since it involves both the filter error, $\tilde{x}_{k|k}$, and the controlled plant state, x_k . It is described by the linear system:

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1|k} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} A - BK & BK(I - L_k C) & 0 \\ 0 & A(I - L_k C) & 0 \\ C(A - BK) & CBK(I - L_k C) & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_{k|k-1} \\ y_k \end{bmatrix} + \begin{bmatrix} -BK L_k(v_k + n_k) + w_k \\ -AL_k(v_k + n_k) + w_k \\ -CBK L_k(v_k + n_k) + Cw_k + v_{k+1} \end{bmatrix}, \quad (11)$$

This system's state covariance matrix is shown as follows,

$$\text{cov} \left(\begin{bmatrix} x_k \\ \tilde{x}_{k|k-1} \\ y_k \end{bmatrix} \right) = \begin{bmatrix} P_k & \Sigma_{k|k-1} & P_k C^T \\ \Sigma_{k|k-1} & \Sigma_{k|k-1} & \Sigma_{k|k-1} C^T \\ CP_k & C\Sigma_{k|k-1} & CP_k C^T + R \end{bmatrix}$$

The Kalman filter recursion for intermittent quantized data, (7-9), yields the controlled state covariance, P_k , recursion

$$\begin{aligned} P_{k+1} = & (A - BK)P_k(A - BK)^T \\ & + (A - BK)\Sigma_{k|k-1}(I - L_k C)^T K^T B^T \\ & + BK(I - L_k C)\Sigma_{k|k-1}(A - BK)^T \\ & + BK(I - L_k C)\Sigma_{k|k-1}(I - L_k C)^T K^T B^T \\ & + BK L_k R_{\text{eff}} L_k^T K^T B^T + Q. \end{aligned} \quad (12)$$

The effect of increasing the measurement noise through quantization is two-fold; the error covariance, $\Sigma_{k+1|k}$, increases and then, as a consequence, the controlled state covariance, P_k , also increases. Additionally, R_{eff} directly drives (12). During periods of packet loss, measurement noise and quantization noise do not affect the system directly, since $L_k = 0$. At times of packet arrival however, the quantization noise effect in increasing the underlying state estimate covariance and controlled state covariance is evident.

2.5 Escape times, conditional escape times and survival times

We have established that for output signals, y_k , within the quantization range $[-\zeta, \zeta]$, both the estimator error and the output feedback controlled system are governed by linear systems driven by white, not-necessarily-Gaussian noises: $\{(7), (8), (9), (10)\}$ for the estimator; and additionally (11)

for the controlled system. This will form the basis for our calculation of escape times.

We make the following definitions.

Definition 2 (Estimator escape time τ_e) For given positive scalar bound G , we define the estimator escape time to be the escape time for the random sequence $\{\tilde{x}_{k+1|k}\}$ starting from the initial condition $\tilde{x}_{1|0} \sim N(0, \Sigma_{1|0})$ and with $\mathcal{D} = \{|\tilde{x}_{k+1|k}|_\infty \leq G\}$. The estimator escape time cdf is the cdf of τ_e given these initial conditions.

Definition 3 (Output escape time τ_o) Given the output quantizer magnitude upper bound, ζ from Assumption 1.5, and given initial state covariance matrix P_1 and state estimate error covariance matrix $\Sigma_{1|0}$, we define the output escape time to be the escape time from $\mathcal{D} = \{|y_k|_\infty \leq \zeta\}$ with initial condition satisfying $y_1 \sim N(0, CP_1 C^T + R_{\text{eff}})$.

The estimator and output escape times are random variables depending on: the bound ζ , the initial covariances $\Sigma_{1|0}$ and P_1 , the realizations of the noise processes x_1 , $\{w_k\}$, $\{v_k\}$, $\{n_k\}$, and the realization of $\{\gamma_k\}$, the sequence of packet arrival successes or failures. Our aim is to compute and to characterize the probability cumulative distribution functions (cdfs) of these escape times as the communication link's arrival probability, P_γ , changes. The results of [20] establish a lower bound on P_γ for the expected value of the conditional estimator covariance to be finite for all time, i.e. $E[\Sigma_{k|k}] < \infty$, which can occur even though the estimate error itself escapes any bounded domain with probability one.

The inclusion of the output escape time problem differs from the analysis of [20, 19], which considers estimator behavior only, and brings us more into contact with works such as [17] which study stochastic stabilization. We note that for the state-estimate-feedback controlled system, the covariance of output y_k depends on the covariance of the controlled state x_k , which in turn depends on the covariance of the state estimation error, $\tilde{x}_{k|k}$.

For specific γ_k sequences, we introduce the following definition:

Definition 4 (Conditional escape time) For bounded domain $\mathcal{D}$ and stochastic process $\{\xi_k\}$ adapted to the sequence $\{\gamma_k\}$, the conditional escape time cdf, $\Phi_e(k)$, is the escape time cdf from $\mathcal{D}$ for a given sequence, $\{\gamma_k\}$, of successful and dropped communications packets.

In particular, we shall identify a special set of conditional escape times associated with unsuccessful communication from time 1 onwards.

Definition 5 (Survival time) We define the survival time cdf for $\{\xi_k\}$, $\Phi_s(k)$, as the conditional escape time cdf for the domain $\mathcal{D}$ with sequence $\{\gamma_k = 0, k = 1, 2, \dots\}$.

3 Escape Time Markov Chain Analysis

This section provides the fundamental theoretical support for these calculations, which later will be applied to examples.

3.1 General Markov Chain model

Following Definition 1, given a closed domain $\mathcal{D} \subset \mathbb{R}^d$ and a stochastic process $\{\xi_k : k = 1, \dots\}$ on $\mathbb{R}^d$, the escape time of ξ_k from $\mathcal{D}$ is described by a Markov chain. This is a general result concerning adapted processes and is not limited to linear Gaussian systems nor to hyperspherical domains.

Theorem 2 (Principal theoretical result [7]) *For stochastic process $\{\xi_k : k = 1, \dots\}$ the random variable*

$$J_{k+1} = \begin{cases} 1, & \text{if } \xi_k \in \mathcal{D} \text{ and } J_k = 1, \\ 0, & \text{otherwise,} \end{cases} \quad (13)$$

is a Markov process and, denoting

$$\Pi_k = \begin{bmatrix} \Pr(J_k = 1) \\ \Pr(J_k = 0) \end{bmatrix},$$

is described by the Markov chain.

$$\Pi_{k+1} = \begin{bmatrix} \alpha_k & 0 \\ 1 - \alpha_k & 1 \end{bmatrix} \Pi_k,$$

where

$$\alpha_k = \Pr(J_{k+1} = 1 | J_k = 1) \quad (14)$$

The escape time of the stochastic process $\{\xi_k\}$ is the value of k when $J_k = 0$ for the first time.

Proof: From the definition (13), J_k satisfies the Markov property.

$$\Pr(J_{k+1} | J_k, \dots, J_1) = \Pr(J_{k+1} | J_k),$$

and so $\{J_k\}$ is described by a Markov chain. The transition matrix of this chain is given by

$$\begin{bmatrix} \Pr(J_{k+1} = 1 | J_k = 1) & \Pr(J_{k+1} = 1 | J_k = 0) \\ \Pr(J_{k+1} = 0 | J_k = 1) & \Pr(J_{k+1} = 0 | J_k = 0) \end{bmatrix}$$

The (1,2)-element of this matrix is zero, because there is no possibility of moving from $J_k = 0$ to $J_{k+1} = 1$. ■

Corollary 1 *For a given $\{\gamma_k\}$ sequence, the corresponding $\{\alpha_k\}$ sequence describes the evolution of the conditional escape time cdf, $\Phi_e(k)$,*

$$\begin{aligned} \Pr(J_{k+1} = 0) &= (1 - \alpha_k)\Pr(J_k = 1) + \Pr(J_k = 0), \\ \Phi_e(k+1) &= (1 - \alpha_k)[1 - \Phi_e(k)] + \Phi_e(k), \end{aligned}$$

or,

$$\Phi_e(k+1) = (1 - \alpha_k) + \alpha_k \Phi_e(k), \quad (15)$$

whence

$$\Phi_e(k) = 1 - \prod_{i=0}^{k-1} \alpha_i. \quad (16)$$

Proof: The final result (16) follows by rewriting (15) as

$$[1 - \Phi_e(k+1)] = \alpha_k [1 - \Phi_e(k)],$$

and recognizing that $1 - \Phi_e(1) = \alpha_0$. ■

3.2 Computation of α_k for linear systems

As developed in Theorem 2, the escape time of stochastic process $\{\xi_k\}$ is governed by a Markov chain and the probability transition matrix is determined by a sole scalar value α_k . For any escape time problem, α_k might be computed as

$$\alpha_k = \frac{\Pr(\xi_{k+1} \in \mathcal{D}, \xi_k \in \mathcal{D}, \dots, \xi_1 \in \mathcal{D})}{\Pr(\xi_k \in \mathcal{D}, \dots, \xi_1 \in \mathcal{D})}.$$

This requires the computation of a $(k+1)n$ -dimensional and a kn -dimensional multivariate Gaussian cdf, as could in the Gaussian case in principle be carried out using `mvncdf` in MATLAB. However, this is problematic due to the growth of dimension of the argument. We now present an alternative recursion for the computation of α_k for linear systems.

Theorem 3 *Consider the time-varying stochastic linear system*

$$\xi_{k+1} = F_k \xi_k + \omega_k, \quad (17)$$

with: ξ_1 possessing a pdf, $f_{\xi_1}(z)$, and $\{\omega_k\}$, independent from ξ_1 , white and with pdfs $f_{\omega_k}(z)$.

Take

$$h_0(z) = f_{\xi_1}(z), \quad (18)$$

and define the sequence of (pdf) functions $\{h_k : \mathbb{R}^n \rightarrow \mathbb{R}_+\}$

$$h_k(z) = \frac{1}{\int_{\mathcal{D}} h_{k-1}(w) dw} \int_{\mathcal{D}} f_{\omega_k}(z - F_k w) h_{k-1}(w) dw. \quad (19)$$

Then

$$\alpha_k = \int_{\mathcal{D}} h_k(w) dw. \quad (20)$$

Proof: This is proven in the Appendix. $\blacksquare$

Theorem 3 permits calculation of $\{\alpha_k\}$ for (17) for any process noise pdfs, even time-varying. In the sequel for comparison with earlier results, we shall limit investigation to either Gaussian or Gaussian-plus-uniform noise

Corollary 2 For linear Gaussian system (17) with initial state $\xi_1 \sim N(0, \rho_1)$ with $\rho_1 > 0$, and $\omega_k \sim N(0, \Omega_k)$, white and independent from ξ_1 , α_k is given by the recursion (18-20) with

$$\begin{aligned} f_{\xi_1}(z) &= \frac{1}{(2\pi)^{n/2} |\rho_1|^{1/2}} \exp\left(-\frac{1}{2} z^T \rho_1^{-1} z\right), \\ f_{\omega_k}(z - F_k w) &= \frac{1}{(2\pi)^{n/2} |\Omega_k|^{1/2}} \exp\left[-\frac{1}{2} (z - F_k w)^T \Omega_k^{-1} (z - F_k w)\right]. \end{aligned}$$

For the same system and initial state but with $\omega_k = v_k + n_k$ where $v_k \sim N(0, R_k)$ and $n_k \sim U(-\zeta/2^b, \zeta/2^b)$ with $\{v_k\}$ and $\{n_k\}$ independent and white, α_k is given the recursion (18-20) with $f_{\xi_1}(z)$ as above and $f_{\omega_k}(z)$ given by $f_{v+n}(z)$ from (5).

In both the estimator and the output escape problems, a given $\{\gamma_k\}$ sequence and initial state estimate covariance determine the associated Kalman gain sequence $\{L_k\}$, which captures the time-variability of the problem. Both cases can be subsumed into the study of the linear system (11) with the corresponding escape set. For example, the corresponding escape set of output escape time $\mathcal{D}$ and initial covariance $\mathcal{C}$:

$$\mathcal{D} = \left\{ \begin{bmatrix} x_k \\ \tilde{x}_{k|k-1} \\ y_k \end{bmatrix} : |y_k|_{\infty} \leq \zeta \right\}; \mathcal{C} = \begin{bmatrix} P_1 \\ \Sigma_{1|0} \\ CP_1 C^T + R \end{bmatrix} \quad (21)$$

The escape set of estimation escape time uses $\{|\tilde{x}_{k|k-1}|_{\infty} \leq G\}$ instead of $\{|y_k|_{\infty} \leq \zeta\}$ in above equation.

For the estimator or for the output, the results of this section permit the calculation of the conditional escape time cdf for any given packet arrival sequence, $\{\gamma_k\}$. Since in this case, the system matrices, F_k , are known.

With $\gamma_k = 0, k = 1, 2, \dots$, the survival time analysis for both the estimator escape and for the output escape with linear systems is a time-invariant analysis, since the Kalman

gain $L_k = 0$. Accordingly, the study of survival time can be explicitly conducted, which provides insight into the processes underlying escape times, which usually require simulation for evaluation.

3.3 Properties of output escape time for linear time-invariant Gaussian systems

Denote by $\Phi_e(t)$ the cdf of the output escape time for the linear time-invariant Gaussian system

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad (22)$$

$$y_k = Cx_k + v_k, \quad (23)$$

with

$$\begin{aligned} u_k &= -K\hat{x}_{k|k}; \quad x_1 \sim N(0, P_1); \\ w_k &\sim N(0, Q); \quad v_k \sim N(0, R). \end{aligned}$$

By Lemma 1 since $\mathcal{D} = \{|y_k|_{\infty} \leq \zeta\}$ is bounded, $\Phi_e(t) \rightarrow 1$ as $t \rightarrow \infty$. For the same γ -sequence (which means the same F_k, ω_k in Theorem 3), Φ_e is a function of t, P_1, Q, R and ζ . From the Theorem 3, we further have the following properties holding for all $t \geq 1$ where the superscript delineates each of two different cases:

$$\begin{aligned} \{P_1^1 \geq P_1^2\} \cup \{Q^1 \geq Q^2\} \cup \{R^1 \geq R^2\} \cup \{\zeta^1 \leq \zeta^2\} \\ \implies \Phi_e(t, P_1^1, Q^1, R^1, \zeta^1) \geq \Phi_e(t, P_1^2, Q^2, R^2, \zeta^2), \end{aligned} \quad (24)$$

The proof of this property appears in the Appendix.

Theorem 4 For the linear time-invariant Gaussian system (22-23), the cdf of the survival time, Φ_s , admits the following implication

$$\begin{aligned} P_1^1 \geq A^n P_1^2 A^{n^T} + A^{n-1} Q A^{n-1^T} + \dots + A Q A^T + Q \\ \implies \Phi_s(t, P_1^1, Q, R, \zeta) \geq \Phi_s(t + n, P_1^2, Q, R, \zeta). \end{aligned}$$

Theorem 4 will be used when studying the effect of quantization errors on the escape time.

4 Escape time computational examples

4.1 Example 1: Estimator survival time

Recall that the survival time is the escape time when no data packets are received at times $k = 1, 2, \dots$. Because Theorem 3 provides an explicit formula for α_k , we study first the survival time of the estimator. This example, while simple, permits revealing comparisons between the escape/survival time analysis and the moment-based methods of [20, 19].

Consider the system

$$A = 1.5, B = 1, C = 1, Q = 0.005, R = 0.001, \zeta = 10, \\ \text{with } \Sigma_{1|0} = AR_{\text{eff}}A^T + Q = 0.00725,$$

which corresponds to the largest possible conditional prediction covariance immediately after receiving a single data sample. We compute the survival time cdf using Theorems 2 and 3 which is a time-invariant system transient analysis yielding explicitly the survival cdf. The probability density function is plotted in Figure 4.

Fig. 4. Probability density function of estimator survival time for Example 1.

The calculation of the density function of τ_s was verified, needlessly, by simulation. Figure 5 displays three sample $\tilde{x}_{k+1|k}$ trajectories of differing survival times: 8, 12, and 19, from a Monte Carlo simulation with initial covariance $\Sigma_{1|0} = 0.00725$. We have selected three trajectories which escaped with positive state-estimate error values. From the

Fig. 5. Three different estimator survival time trajectory samples for Example 1 with initial error covariance $\Sigma_{1|0} = 0.00725$.

problem description, the state-estimate error satisfies,

$$\tilde{x}_{k+1|k} = (1.5)^k \tilde{x}_{1|0} + \sum_{j=1}^k (1.5)^{k-j} w_j,$$

which displays the relative importance of the random terms. Since $w_k \sim N(0, 0.005)$, once the state exceeds roughly 0.2 (three σ) in magnitude it becomes most unlikely that w_k will arise to bring the error back to a small magnitude. Once the three trajectories exceed this value, we see that they all escape with roughly similar behavior, i.e. at a rate of 1.5^k . The difference between the trajectories lies in their residence time in the neighborhood $|\tilde{x}| < 0.2$. Here the disturbance process is as likely to drag the error back towards small magnitudes as it is to increase the magnitude, resulting in some trajectories remaining close to zero for an extended time. This is captured by the Markov analysis.

4.2 Markov versus covariance comparison

Comparing the current Markov results to the covariance calculations of [20], we see that, in both survival time analyses, the conditional covariance is given by

$$\Sigma_{k+1|k} = A^k \Sigma_{1|0} A^{k^T} + \sum_{j=0}^{k-1} A^j Q A^{j^T}.$$

In contrast to the Markov analysis, covariance $\Sigma_{k+1|k}$ may be used to compute directly the probabilities $\Pr(|\tilde{x}_{k+1|k}|_2 \leq \zeta)$ and $\Pr(|\tilde{x}_{k+1|k}|_2 > \zeta)$ using the normal cdf. Whence, the escape probability may be approximated via

$$\Pr(|\tilde{x}_{k+1|k}|_\infty > \zeta) \times \prod_{j=1}^k \Pr(|\tilde{x}_{k+1-j|k-j}|_\infty < \zeta). \quad (25)$$

This calculation may be performed using the stationary expected conditional covariance or using the data-dependent covariance, $\Sigma_{k+1|k}$, derived from the $\{\gamma_k\}$ sequence.

Figure 6 displays three curves computed for Sinopoli's example from [20] in the region where the expected conditional covariance is finite, i.e. $P_\gamma > 0.36$. Here,

$$A = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix}, Q = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}, P_\gamma = 0.4$$

$$\bar{M} = \begin{bmatrix} 67.36 & -0.052 \\ -0.052 & 22.73 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, R = 2.5,$$

$$\zeta = 100.$$

Fig. 6. Comparison of escape time cumulative distribution functions for Sinopoli's Example: unbroken curve – simulation, dash-dot curve – estimate using fixed expected conditional covariance and (25), dashed curve – estimate using time-varying conditional covariance and (25) then averaged over 1000 trials.

This calculation demonstrates that attention solely to the covariance is problematic and, further, that the stationary expected conditional covariance is pessimistic in its implication regarding escape time, while the time-varying covariance is overly optimistic. This corroborates the results of [19], where the expected value of the conditional covariance is replaced by analysis of its probabilistic behavior.

4.3 Example 1 redux: Output survival times with quantized data

We return to the scalar-state Example 1, but now with state-estimate feedback control. The parameters of the controlled system and output signal bound, ζ , are:

$$A = 1.5, B = 1, C = 1, D = 0, K = 1.2, \\ Q = 0.005, R = 0.001, \zeta = 10.$$

We consider two quantization scenarios; 5-bit quantization of y_k , and 16-bit quantization.

Figure 7 depicts the cdfs for 16-bit (blue) and 5-bit (red) for P_γ taking values in $\{0.2, 0.3, 0.4, 0.5\}$. It is noteworthy that the explicitly quantified difference in survival times of the differing bitrates, described by Theorem 4 in the case where $P_\gamma = 0$, is exacerbated when P_γ increases.

Figure 8 shows the variation in escape times for increasing values of the bound, ζ , defining the escape domain $\mathcal{D}$, and for fixed $P_\gamma = 0.2$. The bound ζ takes values in the set $[1, 10, 100, 500]$. This affects the quantization intervals and, in turn, affects: the effective measurement noise, R_{eff} , as in (6); the estimator covariance value $\Sigma_{k+1|k}$; and the initial controlled state covariance, P_1 .

It is apparent that, for small numbers of bits, the quantization error becomes a significant factor when its effect on P_1

Fig. 7. Semilog plot of cumulative distribution functions of output escape times for 5-bit (red) and 16-bit (blue) quantized measurements as the probability of successful packet arrival, P_γ , varies. The order of the plots left-to-right in the center corresponds to the order down the legend.

Fig. 8. Cumulative distribution functions of controlled output escape times for Example 1 with $\zeta = 1, 10, 100, 500$, and with $P_\gamma = 0.2$. Results are shown for 5-bit quantization (red) and 16-bit quantization (blue).

dominates that of the process noise Q . When the initial state covariance P_1 becomes large enough, the dependence of escape time on ζ depends primarily on growth in ζ versus the maximal eigenvalue of A . Thus, for these ζ values and 5-bit quantization, we see little apparent variation in the escape time statistics past $\zeta = 10$, while the 16-bit quantization escape times continue to grow dramatically with ζ .

5 Bitrate assignment and escape time

Communications issues of intermittency and limited bitrate may be combined for joint analysis and design, particularly in the realm of partitioning the available bitrate as a hedge against the deleterious effects of dropped packets. Evidently this is a rudimentary coding question, since one can sacrifice some bits to accommodate error recovery. This represents a departure from [20, 19], whose analysis is restricted to intermittency alone. Although, [19] countenances retransmission of data within each packet, albeit at no cost to the

communication rate. More recently, [13] has studied the application of redundant parallel channels in the amelioration of packet loss effects, again without addressing quantization issues arising due to limited bitrates.

Sinopoli *et al.* [20] assume that the measurement matrix $C = I$ in (2) while [19] assumes that each packet contains r successive measurements, where r is the least value for which

$$\text{range} \left[C^T \ A^T C^T \ \dots \ A^{(r-1)T} C^T \right] \supset \text{range} \ A^{rT}.$$

Further, these data packets are not bitrate limited. In either case, the arrival of any single packet suffices for reconstruction of the state estimate with conditional covariance less than a specific bounded quantity derived from the Information Filter, see [1].

With a limited bitrate and $r > 1$, the assignment of bits within a measurement packet to retransmitted data comes at a cost in terms of the state estimate error, because multiply arriving copies of data carry no information and the associated bits might better have been used for the new data. We next analyze by simulation the effect of such bitrate assignment on escape time.

Suppose that the bits at any time are assigned to the most recent data and also to some fraction of the earlier data, as is illustrated in Figure 9. This redundancy in data transmis-

Fig. 9. Bit allocation scheme. Here 3 of 8 bits at time k are assigned to y_{k-1} and 5 bits to y_k .

sion should provide some protection from packet loss at the expense of estimation accuracy. This impact can be subtle, since the reduction in bits for y_k affects both the limiting filtered estimate and controlled state covariances.

Our example takes

$$A = \begin{bmatrix} 0 & 1.5 \\ 1.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0,$$

$$Q = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, R = 0.01, \zeta = 5, K = [-1.1, 1.66].$$

Case A: We take 8 bits for y_k at each time k . Then the appropriate initial covariances are

$$\Sigma_{0|0} = \begin{bmatrix} 0.010 & 0 \\ 0 & 0.0073 \end{bmatrix}; P_0 = \begin{bmatrix} 1.027 & 1.353 \\ 1.353 & 2.361 \end{bmatrix}.$$

Case B: We take 4 bits each for y_{k-1} and for y_k with corresponding

$$\Sigma_{0|0} = \begin{bmatrix} 0.075 & 0 \\ 0 & 0.219 \end{bmatrix}, P_0 = \begin{bmatrix} 2.404 & 3.099 \\ 3.099 & 5.459 \end{bmatrix}.$$

Escape time cdfs are shown in Figure 5 for the case where $P_\gamma = 0.3$. We see that, in this case, the retransmission strategy serves to extend the time for escape attributable to nefarious sequences $\{\gamma_k\}$ at the cost of increasing the likelihood of early escape ascribed to the concomitant increase in initial covariances.

Fig. 10. Output escape time cdfs with single transmission (Case A) and split bitrate signal retransmission (Case B) when $P_\gamma = 0.3$

6 Comments and conclusions

We have presented an approach to the study of state estimation and state-estimate feedback stabilization with intermittent and quantized measurements. This is based on escape time analysis and computation, which is compared to earlier works in estimation and stabilization with differing descriptions of the properties – finite expected conditional covariance and mean-square stabilization – and of the communication system – intermittent but exact and quantized but certain. The central result is that the escape time can be described by a Markov chain, which in the linear case is easily computed. This yields much more precise evaluation of behavior than covariance calculations, even in the Gaussian case. Our analysis has included the treatment of quantization effects in output escape time problems and shown that bitrate choice can have a profound effect on the output escape time properties of a controlled system.

Coda & Acknowledgement

The authors are grateful to the perspicacious anonymous reviewers who pointed out important technical errors in early

versions of this paper. The thrust of these problems was to replicate in the escape time framework the resetting of covariance-based analyses once certain sequences of successful packets arrive. In the Kalman filtering context, this is used to guarantee the return to an upper bounded conditional covariance matrix – the actual bound follows from the Information Filter formulation of the problem. This resetting permits the separation of the analysis of future behavior from that prior to the reset. In the escape time formulation this separation is no longer possible and it is instructive to understand why.

In the Gaussian case, one (including the authors) might be expected to use the following relation following from (14).

$$\begin{aligned}\alpha_k &= \Pr(J_{k+1} = 1 \mid J_k = 1), \\ &= \Pr(J_{k+1} = 1 \mid J_k = 1, J_{k-1} = 1), \\ &= \frac{\Pr(J_{k+1} = 1, J_k = 1 \mid J_{k-1} = 1)}{\Pr(J_k = 1 \mid J_{k-1} = 1)}.\end{aligned}$$

This latter quantity is familiar from Kalman filtering and computations such as

$$\Pr(x_{k+1} | y_k, y_{k-1}, \dots, y_1) = \frac{\Pr(x_{k+1}, y_k | y_{k-1}, \dots, y_1)}{\Pr(y_k | y_{k-1}, \dots, y_1)}.$$

However for escape times, the conditioning is not on the specific value taken by a signal such as y_k . Instead, the conditioning is over the residence of $x_k \in \mathcal{D}$ and the probabilities above need to be computed as integrals over all of $\mathcal{D}$. The finite-dimensionality of the Kalman filter is lost since, even though (x_{k+1}, x_k) are jointly Gaussian, the conditional probability $\Pr(x_{k+1} | x_k \in \mathcal{D})$ is not Gaussian. This is evident in the precise calculations of Theorem 3.

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7 Appendix

7.1 Proof of Lemma 1

Without loss of generality take $\mathcal{D}$ to be a hypersphere in $\mathbb{R}^n$, i.e. $|x|^2 \leq r$ for some $r > 0$. If this is not the case then, since $\mathcal{D}$ is compact, we can replace $\mathcal{D}$ by another compact set $\mathcal{D}^+$ which is a hypersphere containing $\mathcal{D}$ and perform the same analysis. Escape from $\mathcal{D}^+$ certainly implies escape from $\mathcal{D}$.

At time k with state x_k and control u_k , both of which are independent from w_k under the conditions of the lemma, the condition for x_{k+1} to lie outside $\mathcal{D}$ is that

$$Ax_k + Bu_k + w_k \in \mathcal{D}^c,$$

the complement of $\mathcal{D}$. Since $\mathcal{D}$ is a hypersphere centered at the origin and w_k is zero mean Gaussian

$$\Pr(w_k + v \in \mathcal{D}) \leq \Pr(w_k \in \mathcal{D}) = \beta < 1,$$

for any vector v . This argument can be extended to multiple steps as follows.

$$\begin{aligned} & \Pr(w_k + v_k \in \mathcal{D}, w_{k+1} + v_{k+1} \in \mathcal{D}) \\ &= \Pr(w_{k+1} + v_{k+1} \in \mathcal{D} \mid w_k + v_k \in \mathcal{D}) \Pr(w_k + v_k \in \mathcal{D}), \\ &\leq \beta \Pr(w_k + v_k \in \mathcal{D}), \\ &\leq \beta^2. \end{aligned}$$

So the probability of escape by time k is bounded by β^k and, by the Borel-Cantelli Lemma, we have the probability of no escape is zero. $\blacksquare$

7.2 Proof of Theorem 3

The proof hinges on the careful application of the Markov property. Denote by p the general pdf where the argument specifies the pdf and define

$$h_k(\xi_{k+1}) = p(\xi_{k+1} \mid J_k = 1), \quad (26)$$

$$= \frac{p(\xi_{k+1}, J_k = 1)}{\Pr(J_k = 1)}. \quad (27)$$

Now note that the numerator

$$\begin{aligned} & p(\xi_{k+1}, J_k = 1) \\ &= \int_{\xi_k \in \mathcal{D}} p(\xi_{k+1} \mid \xi_k, J_{k-1} = 1) p(\xi_k, J_{k-1} = 1) d\xi_k, \\ &= \int_{\xi_k \in \mathcal{D}} p(\xi_{k+1} \mid \xi_k) p(\xi_k, J_{k-1} = 1) d\xi_k, \\ &= \Pr(J_{k-1} = 1) \int_{\xi_k \in \mathcal{D}} p(\xi_{k+1} \mid \xi_k) p(\xi_k \mid J_{k-1} = 1) d\xi_k, \\ &= \Pr(J_{k-1} = 1) \int_{\xi_k \in \mathcal{D}} f_{\omega_k}(\xi_{k+1} - F_k \xi_k) h_{k-1}(\xi_k) d\xi_k, \end{aligned} \quad (28)$$

where f_{ω_k} is the probability density function of ω_k . [The Markov property is invoked in replacing $p(\xi_{k+1} \mid \xi_k, J_{k-1} = 1)$ by $p(\xi_{k+1} \mid \xi_k)$.] Substituting (28) into (27) yields

$$\begin{aligned} & h_k(\xi_{k+1}) \\ &= \frac{\Pr(J_{k-1} = 1)}{\Pr(J_k = 1)} \int_{\xi_k \in \mathcal{D}} f_{\omega_k}(\xi_{k+1} - F_k \xi_k) h_{k-1}(\xi_k) d\xi_k, \\ &= \frac{1}{\alpha_{k-1}} \int_{\xi_k \in \mathcal{D}} f_{\omega_k}(\xi_{k+1} - F_k \xi_k) h_{k-1}(\xi_k) d\xi_k. \end{aligned}$$

Finally, integrating (26) yields

$$\begin{aligned} \alpha_k &= \Pr(\xi_{k+1} \in \mathcal{D} \mid J_k = 1), \\ &= \int_{\xi_{k+1} \in \mathcal{D}} p(\xi_{k+1} \mid J_k = 1) d\xi_{k+1}, \\ &= \int_{\xi_{k+1} \in \mathcal{D}} h_k(\xi_{k+1}) d\xi_{k+1}. \end{aligned}$$

7.3 Proof of the property (24)

Consider the time-varying, linear, Gaussian system (11) derived from (22-23) and written in the form of (17) with zero-mean initial state. That is,

$$\xi_{k+1} = F_k \xi_k + \omega_k,$$

with $\xi_0 \sim \mathcal{N}(0, \mathcal{P})$ and $\omega_k \sim \mathcal{N}(0, \Upsilon_k)$. Further, consider escape from the region $\mathcal{D} = \{|D\xi_k| \leq g\}$ for some constant matrix D . At time t , escape of this system coincides with

$$\Xi_t = \begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_t \end{bmatrix} \notin \mathcal{D}^{t+1} = \overbrace{\mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}}^{t+1 \text{ times}},$$

and Ξ_t is Gaussian and zero mean. Now using *increase* to mean *does not decrease*, if either $\text{cov}(\xi_0)$ increases or $\text{cov}(\omega_k)$ increases then the covariance of ξ_t increases and escape from the set $\mathcal{D}^t$ as defined above increases. By the same token, if the parameter g decreases then the escape probability increases. This proves (24).