Graph-Theoretic Analysis of Network Input-Output Processes: Zero Structure and its Implications on Remote Feedback Control [★]

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Abstract

The control of dynamical processes in networks is considered, in the case where measurement and actuation capabilities are sparse and possibly remote. Specifically, we study control of a canonical network dynamics, when only one network component's state can be measured and only one (in general different) component can be actuated. To do so, we characterize the finite- and infinite- zeros of the resulting SISO system in terms of the graph topology. Using these results, we establish graph-theoretic conditions under which there are zeros in the closed right-half plane. These conditions depend on the length, strength, and number of the paths from the component where the input is applied to the component where the measurements are made. Then, we present the implications of these conditions on the controller design task focusing in stabilizations/destabilization of network processes under static negative feedback.

Key words: Interconnected Systems, Graph Theory, Non-minimum Phase Systems,

1 Introduction

Network synchronization and diffusion models are used to capture such diverse processes as vehicle coordination, biochemical reactions, and sensor-fusion algorithms [28]. The temporal dynamics of these models have been extensively studied from a graph-theoretic perspective. First, emergence of synchronization has been shown under broad connectivity conditions, i.e. the models have been shown to have a stable manifold wherein all the network components' states are identical [4,29]. Further results have been developed that tie performance characteristics (e.g., the settling rate) to features of the network's graph. As synchronization models have found wider application in engineering contexts, their design and control have also been of significant interest.

Many of these studies consider decentralized control of multiple autonomous but communicating agents (e.g., vehicles), which yields a closed-loop dynamics that is a network synchronization process. In complement, several recent studies have considered topology design to shape the performance of synchronization processes [21,2,3,27].

While the literature has focused on emergence and design of whole-network behaviors such as synchronization, there is a growing need to understand input-output dynamics and feedback regulation of established network processes, when measurement and actuation are available at only a few network components. This need for an input-output analysis partially stems from challenges in security and vulnerability analysis of infrastructures and other complex dynamical networks [22,23,26,13,5,18,17]. In these applications, an adversary can typically only make limited measurements and actuations of the dynamics, but may be able to initiate a significant propagative impact across the network [22,23]. By the same token, system operators may often have limited measurement and actuation capabilities in reacting to threats/disturbances, and their ability to mitigate the wide-area threats via local feedback is of

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interest. The input-output dynamics of network processes are also germane to management and resource allocation problems in large-scale networks, where limited control resources must be placed to shape global network dynamics. A key need in these various application domains is to understand the implications of the network's graph topology on the input-output dynamics and specifically its zeros, to achieve simple insights into propagative impacts and enable control design.

Recently, several studies have begun to study the input-output dynamics of network synchronization and spread processes from a graph-theoretic perspective. Our work is closely aligned with the study of Briegel et al, which characterizes the zeros of a single-input-single-output (SISO) system defined on a consensus (synchronization) process [7]. The authors focus particularly on symmetric unweighted network topologies, and give bounds on finite-zero locations and conditions for the presence of right-half-plane zeros. Meanwhile, our previous work pursues a structural decomposition of an input-output dynamics imposed on a synchronization process [3], and uses this decomposition to achieve simple graphical characterizations of the zeros. Motivated by vulnerability-analysis goals, control theorists have also studied robustification of synchronization processes via feedback [26], characterized disturbance propagation [13], and highlighted linkages between graph connectivity and network robustness (via the presence of non-trivial zero dynamics) [5,18,17].

This short paper is concerned with the input-output dynamics of a class of continuous-time linear network processes defined on a general (directed, weighted) graph, which is actuated at a single network component and measured at another component (see Section 2). For this SISO model, a full characterization of the infinite- and finite- zeros in terms of the network's graph topology and the actuation/measurement locations is undertaken (Section 4), using a structural decomposition for the input-output dynamics (reviewed in Section 3). A significant result is that networks with weak short paths as well as alternative long strong paths between the input and output have non-minimum-phase zeros. Some implications on feedback control of the network dynamics are briefly discussed, particularly focusing on destabilization through remote feedback.

2 Problem Formulation

We are concerned about the input-output behavior of a dynamical-network process that is actuated at a single network component, and measured at a single component (which in general may be remote from the stimulation location). Formally, a network with n components, labeled $1, 2, \dots, n$, is considered. Each component is assumed to have a scalar state $\tilde{x}_i$ associated with it. These

states evolve according to the differential equations:

$$\dot{\tilde{x}} = A\tilde{x} + e_i\tilde{u} \quad (1)$$

where $\tilde{x} = [\tilde{x}_1 \dots \tilde{x}_n]'$ is the full state of the network, e_i is 0-1 indicator vector with i^{th} entry equal to 1, $\tilde{u}$ is a scalar input signal at a single network component i , and the state matrix A of the network dynamics is called the *graph matrix*. We assume that the off-diagonal entries of the state matrix A are nonnegative, and the diagonal entries are negative and satisfy $A_{i,i} \leq -\sum_{j=1, j \neq i}^n A_{i,j}$. Dynamical models of this form are commonly used to represent both natural and engineered network processes (e.g., circuit dynamics, fluid-flow systems, satellite alignment). The special case that $A_{i,i} = -\sum_{j=1, j \neq i}^n A_{i,j}$ has been particularly well studied, as a canonical model for synchronization or consensus or diffusion. We stress that the matrix A need not be symmetric. The state of one component, say component n WLOG, is measured. Thus, the observation or output $\tilde{y}$ of the dynamical network is given by:

$$\tilde{y} = e_n'\tilde{x}. \quad (2)$$

Equations (1) and (2) together specify a SISO network process.

A weighted digraph G with n vertices is associated with the network dynamics, where each vertex $i = 1, 2, \dots, n$ in the graph corresponds to the network component i . Formally, an arc (directed edge) is drawn from vertex i to vertex j in the graph (i, j distinct) if and only if $A_{j,i} \neq 0$, and is assigned a weight of $A_{j,i}$. The vertices corresponding to the input and output network components are referred to as the input and output vertices. The state matrix $-A$ can be viewed as (the transpose of) a Laplacian or grounded Laplacian matrix associated with the graph, per our definition.

Our goal is to characterize the zeros of the SISO system (1) and (2) in terms of the digraph G and the input/output locations. Specifically, we seek graph-theoretic conditions under which the infinite zeros (asymptotes of the positive root-locus branches) are in the left- or right- half plane. Also, graph-theoretic characterizations of the the number and locations of the finite zeros are sought. These results then imply myriad limits on feedback control (e.g., on perfect tracking) imposed the graph topology, as we will discuss briefly. As a particular application, we will draw on graph-theoretic conditions for nonminimum-phase dynamics to determine when static negative feedback $\tilde{u} = -K\tilde{y}$ yields instability.

3 Background

In [3], we used the special coordinate basis (SCB) for linear systems [24] to obtain some preliminary structural results on the zeros of the SISO network model ((1) and (2)), which are foundational to the graph-theoretic results developed here. The SCB is a convenient tool for the graph-theoretic analysis of zeros, because it provides an explicit matrix-algebraic characterization of a system's zero structure. Specifically, the SCB expresses a linear system as two feedback-interconnected subsystems: 1) an integrator chain that is directly driven by the system input and directly impacts the output, and 2) a zero dynamics that is neither directly driven by the input nor directly affects the output. The transformation of (1) and (2) into the SCB and resultant structural analyses are summarized in the following list [3]. The development uses standard terminology related to finite and infinite zeros, and zero dynamics, please see [24,8,14].

1) The relative degree (number of infinite zeros) is given by $n_d = d + 1$, where d is the distance from the input to the output vertex in G (see also [7,20]). In the SCB formulation, the states associated with the vertices in the shortest directed input-output path form the chain of n_d integrators. Let us call this path the *special input-output path*.

2) The dimension of the zero dynamics (number of finite zeros) is $n_a = n - d - 1$. The states of the zero dynamics can be defined via a transform of the states corresponding to vertices that are not on the special input-output path. We define a set V_1 containing these n_a vertices. We also use the notation G_1 for the induced subgraph of G on V_1 . WLOG, the vertices on the special input-output path are labeled $n - d, n - d + 1, \dots, n$, where vertex $n - i$ is at a distance i to the output, while the vertices in V_1 are labeled $1, 2, \dots, n_a$.

3) The network's finite-zero dynamics is given by:

$$x_0 = A_{aa}x_0 + \left(\sum_{i=0}^{n_d-1} A_{aa}^{n_d-i-1} A_{nad} Z_{nd}^{-1} e_{n_d-i} \right) \tilde{y} \quad (3)$$

where $A_{aa} = A_{n_a} - \Delta$ and $\Delta = A_{nad} Z_{nd}^{-1} Z_{nad}$. The matrix A_{n_a} is a principal submatrix of the A formed by the rows and columns corresponding to the vertices in V_1 . The matrix A_{nad} is an off-diagonal submatrix of A , while Z_{nd}^{-1} and Z_{nad} can be computed explicitly in terms of powers of the graph matrix (see [3] Appendix A2); due to space constraints, we omit the full expressions here but present them as needed in the proofs (see Section Appendix). Since A_{n_a} specifies the graph G_1 , we refer to A_{n_a} as the *reduced graph matrix*. The eigenvalues of the zero-dynamics state matrix A_{aa} are the finite zeros of the process [24].

4) We can view the zero-dynamics state matrix A_{aa} as a perturbation of the matrix A_{n_a} by the matrix Δ . The key idea is that the zero structure is specified by the reduced graph G_1 but with a modification arising via its connection to the special input-output path. The perturbation Δ is a sparse matrix whose nonzero entries are identifiable by the graph structure [3]. Specifically, $\Delta_{ij} = 0$ ($\{A_{aa}\}_{i,j} = \{A_{n_a}\}_{i,j}$) unless the following two conditions are satisfied:

- a) There is an arc (directed edge) from a vertex in the special input-output path (excluding the output) to the vertex $i \in V_1$.
- b) There is a directed path from the vertex $j \in V_1$ to the output vertex whose length is less than or equal to $d - d_i + 1$, where d_i is the distance from the input to the vertex $i \in V_1$ satisfying condition 1.

In addition to the above structural results, our development here draws on: 1) analyses of essentially-nonnegative matrices [6] and 2) a root-locus perspective. Precisely, the eigenvalues of $A - K e_i e'_n$, which are equivalently the poles of the network's closed-loop dynamics upon application of a static negative feedback with gain K , can be traced as a function of the scalar K (for $K > 0$) using a root locus. The asymptotes of the root-locus branches, corresponding to large K , identify the finite and infinite zeros of the open-loop network (1) and (2).

4 Results

Graph-theoretic characterizations of the infinite and finite zeros are presented. Implications on control, including on stabilization and destabilization through static feedback, are discussed. Finally, an example is given. Graph-theoretic terminologies are introduced as needed.

4.1 Graph-Theoretic Results on Zeros:

We characterize whether the infinite and finite zeros of the closed-loop dynamics¹ are in the open left-half plane (OLHP) or the closed right-half plane (CRHP), in terms of the network's graph G . The case where the input and output are collocated ($n_d = 1$) is addressed first (Lemma 1). The infinite zero structure is then determined for non-collocated input and output ($n_d \geq 2$), in Lemma 2 and Theorems 3 and 4. We then characterize the location of the finite zeros in the non-collocated case, focusing on determining the presence or absence of nonminimum-phase zeros (Theorems 5 and 6). To simplify the presentation, proofs of are relegated to Appendix A.

¹ To simplify the presentation, we use the non-standard terminology that infinite zeros are in a particular half plane, to delineate that the corresponding branches of the root locus asymptote in that half plane.

For the case with collocated input and output ($n_d = 1$), the following Lemma 1 verifies that the finite and infinite zeros of the network are necessarily in the OLHP:

Lemma 1 *Consider the dynamical network (1) and (2). If the input and output are collocated ($n_d = 1$), then the $n - 1$ finite zeros of the dynamical network are in the OLHP. Furthermore, the dynamical network has a single infinite zero, which is also in the OLHP.*

Remark 1 *The lemma verifies that the closed-loop will be stable upon high-gain static negative feedback. In fact, it can be shown that the closed-loop dynamics are stable for any static negative feedback gain. This result is congruent with the one presented by Briegel et al [7] for undirected graphs, but we require a different proof approach (see [1]).*

When the input and output are not collocated ($n_d \geq 2$), the dynamical network in general may have both finite- and infinite- zeros in the left half plane. Next, an analysis of the infinite zeros is undertaken for non-collocated input and output. Specifically, we show that the dynamical network will always have infinite zeros in the CRHP, if the input and output vertices are a distance of at least two apart ($n_d \geq 2$), see Lemma 2. When the input and output are adjacent ($n_d = 2$), the dynamical network may or may not have right half plane infinite zeros. For this case, we develop conditions under which zeros are guaranteed to be in the OLHP or CRHP, when the edge between the input and output is sufficiently weak (Theorem 3) or strong (Theorem 4).

Lemma 2 *Consider the dynamical network in (1) and (2), and assume that the distance between the input vertex and output vertex is at least 2 ($n_d \geq 2$). Then the dynamical network has infinite zeros in the CRHP.*

Theorem 3 *Consider the dynamical network in (1) and (2) with adjacent input and output (relative degree $n_d = 2$). Denote the weight of the edge between the input and output as $\epsilon > 0$. Then, the infinite zeros are in the CRHP for all sufficiently small ϵ (i.e. for all $0 < \epsilon < f$ for some threshold $f < \infty$), if and only if there is an alternative input-output path whose length is 2.*

Theorem 4 *Consider the dynamical network in (1) and (2) with adjacent input and output (relative degree $n_d = 2$). Denote the weight of the edge between the input and output as $\delta > 0$. Then, for all sufficiently large δ (i.e. for all $\delta > f$ for some threshold $0 < f < \infty$), the infinite zeros are in the OLHP.*

For adjacent input and output ($n_d = 2$), we observe that the presence of alternative directed paths from the input to the output and weak/strong links in the special input-output path play an important role in whether or not the infinite zeros are in the left- or right- half plane.

Next, the locations of the dynamical network's finite zeros are characterized in terms of the graph topology, for non-collocated input and output. We focus particularly on whether or not the dynamical network has finite zeros in the CRHP (i.e. nonminimum-phase zeros). In [3], the finite zeros were shown to be in the OLHP, in the case that there is only one path from the input to the output. This result depended on showing that the eigenvalues of zero-dynamics state matrix A_{aa} are the same as those of the reduced graph matrix, based on the graphical analysis of the perturbation matrix Δ summarized in Section 3 (list item 4). Here, several sufficient conditions for the presence and absence of nonminimum-phase zeros are developed, when there are multiple paths between the input and output. The essential insight that we obtain is that the dynamical network will have nonminimum-phase zeros, if the shortest path from the input to output is weak while longer input-output paths are stronger. Developing this result requires several analysis steps as well as further notation.

First, we show that vertices that are not part of any paths from the input to output can be ignored in deciding whether or not the network is minimum phase. In particular, we argue that the finite zeros can be partitioned into two sets: 1) the eigenvalues of the reduced graph matrix corresponding only to these irrelevant vertices (which are necessarily in the OLHP) and 2) a set which only depends on the subgraph consisting of input-output paths/walks.

To develop the result, we partition the set of vertices V_1 (those not on the special input-output path) in three disjoint subsets V_o, V_s, V_p as follows. First, the set V_o contains the vertices that are only directly connected to the output. That is, any path between the input and these vertices passes through the output. Second, the set V_s contains the vertices are only directly connected to the input. That is, any path between the output and these vertices passes through the input. Third, V_p contains the remaining vertices in V_1 , i.e. those that are on input-output paths/walks other than the special input-output path. Every vertex in V_p has paths/walks to both input and output without involving any vertex from either V_o or V_s . It is important to clarify that edge directions are not considered (i.e., edges are assumed undirected) in these partition definitions; all other analyses in the article, however, do encode edge directions. We also note that the vertices in the sets V_o, V_s, V_p are mutually disconnected in G_1 . Additionally, we find it convenient define the matrices A_o, A_s, A_p to be the principal submatrices of A which contain only the rows and columns corresponding to the vertices in V_o, V_s , and V_p , respectively. We also define the matrix Δ_{pp} , which contains the rows and columns of the perturbation matrix Δ corresponding to the vertices V_p .

The following result verifies that some of the finite zeros are eigenvalues of A_o and A_s (which are necessarily in

the OLHP), while the remaining are eigenvalues of a perturbation of the principal submatrix of A_p .

Theorem 5 *The finite zeros of the dynamics (1) and (2) are the union of the eigenvalues of the principal submatrices A_o and A_s of the state matrix, and the eigenvalues of the perturbed submatrix $A_{aa_p} = A_p - \Delta_{pp}$. Consequently, the CRHP finite zeros of the dynamical network (if any) are eigenvalues of A_{aa_p} .*

The theorem clarifies that CRHP finite zeros only arise as eigenvalues of A_{aa_p} . For that reason, from here on we focus on the part of the dynamical network that involves the vertices in the special input-output path and V_p , which together completely specify A_{aa_p} . We demonstrate that the presence of CRHP finite zeros depends on the relative strengths of the special input-output path and the other paths from the input to the output (which traverse V_p). In particular, we show that, if the special input-output path has a weak link while all other paths are sufficiently long, then the network dynamics will have a CRHP finite zero.

We begin with some terminology. Let $d_{j,o}$ be length (number of edges) of the directed path from a vertex $j \in V_1$ to the output vertex in G , and $d_{i,j}$ be the length of the directed path from the vertex i to j , both in V_1 , in graph G_1 . A pair of vertices (i, j) in V_1 is said to be a feasible pair, if:

- (1) There is a link (directed edge) from a vertex in the special input-output path (excluding the output) to i . That is, the vertex i is adjacent to the special input-output path;
- (2) There is a directed path from j to the output vertex whose length is less than or equal to $d - d_i + 1$, or equivalently $d_i + d_{j,o} \leq d + 1$; and
- (3) $d_{i,j} < \infty$, i.e. there is a directed path from i to j in G_1 .

It turns out that only the paths between feasible pairs impact the characteristic polynomial of A_{aa_p} and hence affect the location of the finite zeros.

We use the terminology *decisive pair* for a feasible pair (i, j) that has the minimum distance $d_{i,j}$ between the two vertices in G_1 , and refer to a minimum directed path between the decisive pair as the *decisive path*. (If there are more than one of these, we can choose any.) We also refer to the vertex i from which the decisive path originates, i.e. vertex i from the decisive pair (i, j) , as the *decisive vertex*. In Fig. 1, we see that the feasible pairs are: (1, 5), (1, 6) and (4, 6), the decisive pair is (4, 6), and the decisive vertex is 4.

The following theorem presents conditions under which the dynamics has finite zeros in the CRHP, in terms of the length of the decisive path and the location of a weak link with respect to the decisive vertex.

Fig. 1. Decisive vertex and decisive path

Theorem 6 *Consider the dynamical network in (1) and (2). Assume that the input and output are not collocated, i.e. $n_d \geq 2$. Also, let us consider a single edge on the input-output path whose weight is $\epsilon > 0$, and let $d_{m,o}$ be the distance from the vertex m incident to this edge (on the output side) to the output. Then, the finite zeros are in the CRHP for all sufficiently small ϵ (i.e. for all $0 < \epsilon < f$ for some threshold $f < \infty$), if the following conditions are satisfied:*

1. The distance from the input to the decisive vertex i is $d_i \leq d_m$.
2. The distance from the vertex j paired with the decisive vertex i , to the output is $d_{j,o} \leq d_{m,o} + 1$.
3. The length of the decisive path is at least 2.

Theorem 6 defines conditions on the network's graph such that the network dynamics have CRHP finite zeros. In particular, the presence of a sufficiently weak link in the special input output path, and the condition that alternative directed paths from the input to the output are all long and strong, together yield CRHP zeros. This result is complementary to the one presented in [7], which studies the impact of cycles on the finite-zero locations for unweighted/undirected network graphs. In contrast, Theorem 6 shows that nonminimum-phase zeros may arise from varied edge weights on multiple directed paths between the input and output.

4.2 Implications on Control

The implications of a linear system's zero structure in its control have been widely studied. These classical results, together with the connections between graph structure and zeros developed here, yield graph-theoretic insights into the control of network dynamics. Here, we briefly highlight a few simple implications of this sort, focusing particularly on the consequences of remote proportional feedback.

Proportional feedback models with (possibly remote) localized measurement and actuation are approximative of various regulation processes in large-scale networks. Of interest to us, cyber- attacks to network decision-making processes (such as consensus processes) can often be

modeled as a proportional feedback that serves to manipulate the dynamics. Remote proportional-feedback models are also representative of wide-area regulatory mechanisms in infrastructure networks, for instance disease management via quarantine and special control schemes in the electric power grid. Concretely, let us imagine application of a static output feedback to the SISO network process (1) and (2): $\tilde{u} = -K\tilde{y} + \tilde{u}$, where $-K$ is a scalar negative feedback gain and $\tilde{u}$ is a reference signal. From classical control theory, increasing the gain magnitude K typically invokes a trade-off, toward lower steady-state error and faster rise, but increased oscillation and perhaps instability. The graph-theoretic analyses developed here show the implications of the network graph and measurement/actuation locations in such control. In particular, collocated input and output permits negative proportional control without instability (Lemma 1). Meanwhile, a high gain yields instability if the input and output are two or more hops away (Lemma 2), at its essence due to the propagation delay imparted by the distance. If the input and output are adjacent, instability may or may not result, depending on the relative strengths of short and long paths between the input and output (Theorems 3, 4, and 6); however, a high gain will yield complex poles and cause oscillation in this case. Also, the input-output dynamics exhibits special features associated with finite nonminimum-phase zeros (e.g., undershoot), whenever the input and output have short weak paths and long strong paths between them. This is a consequence of fast weak propagation of signals from input to output combined with the strong delayed propagation, which also causes instability upon high-gain negative feedback.

A system's zero structure is also well-known to impose several broader limitations on control design. The presence or absence of nonminimum-phase zeros modulates: 1) the ability to achieve perfect regulation/tracking under transient response requirements, 2) error bounds in state estimation, 3) disturbance rejection performance, and the 4) existence of model reference adaptive control (MRAC) [19,15,11,16], among other consequences. For instance, the performance of the optimal linear regulator (or optimal linear estimator) depends on the locations of the open-loop zeros [15], and their relative location to the poles [9]. Nonminimum-phase zeros are also known to increase system sensitivity over some frequency ranges, complicating stabilization under disturbance and noise [12]. Noting these relationships, we see that our work then has identified graph structures that complicate controller design. These graph-theoretic implications bear on control problems related to network processes, including malicious control by an adversary, and deliberate regulation of transient and steady-state responses.

4.3 Example

We consider a network with 10 components that evolves according to (1) and (2), see Fig. 1. The arcs' weights in the graph G are on the order of 0.1, while the arc's weight between vertex 8 and 9 is comparatively small. For this example, the decisive vertex is vertex 4 and the length of the decisive path is 2 (see page 4). Thus, condition 3 of Theorem 6 is satisfied. Additionally, from the fact that $d_{m=9,o} = 1$, $d_{m=9} = 2$, $d_{i=4} = 2$, and $d_{j=6,o} = 2$, conditions 1 and 2 are also satisfied ($d_4 \leq 2$ and $d_{6,o} \leq 2$). From Theorem 6, for ϵ small enough, the network's dynamics has finite zeros in the CRHP. For example, for $\epsilon = 10^{-5}$, has two CRHP zeros at $\lambda_{1,2} = 2.19 \pm 4.32i$. To give some intuition into the result, let us present the zero-dynamics state matrix:

$$A_{aa} = \begin{bmatrix} -0.2 & 0.1 & & & -\mathbf{3.7 \times 10^3} & \mathbf{2.2 \times 10^3} \\ 0.12 & -0.4 & 0.15 & & 0.13 & \\ & 0.1 & -0.33 & & 0.23 & \\ & & & -0.25 & 0.12 & -\mathbf{1.9 \times 10^3} \\ & & 0.1 & 0.2 & -0.3 & \\ & & & & 0.32 & -0.42 \end{bmatrix}$$

where the bolded entries are perturbed from the reduced graph-matrix values A_{n_a} (i.e. these entries in Δ are nonzero). The perturbation results in oppositely-signed off-diagonal entries which yield CRHP zeros.

We also study control via static negative feedback for this example. The closed-loop poles are traced using a root locus (Fig. 2). The root-locus identifies the infinite zeros (asymptotic branches) and finite zeros. Increasing the feedback gain leads to oscillation and then instability, due to the presence of nonminimum phase zeros.

Fig. 2. Root-locus for the poles of the network dynamics (Fig. 1) when the weak edge weight is $\epsilon = 10^{-5}$.

5 Conclusions

We have characterized the infinite and finite zeros of a SISO network process in terms of its underlying (asym-

metric) graph topology and input/output locations. A key outcome is the development of conditions for the presence of nonminimum-phase zeros, in terms of the lengths and strengths of paths between the input and output. Implications of the graph-theoretic analyses on control of the network dynamics are briefly discussed. A particular focus is on remote regulation/manipulation of the dynamics via (negative) proportional feedback: in general, larger distance and presence of alternative strong paths between input and output are seen to complicate regulation, and allow for destabilization. Other implications, such as in exact disturbance rejection, are also briefly highlighted.

Appendix

Because of space limitations, we retain only the proof of Theorem 6, see [1] for the remaining proofs. We point the reader to literature on nonnegative-matrix perturbation analysis and advanced eigen-analysis techniques, for background relevant to the proofs [6,25,10].

A1. Proof of Theorem 6:

To prove Theorem 6, we first find it convenient to present an auxiliary lemma (see [1] for the proof).

Lemma 7 *Let $P_{A_{n_a}}(\lambda)$ be the characteristic polynomial of the reduced graph matrix A_{n_a} and $P_{A_{aa}}(\lambda)$ the characteristic polynomial of the invariant zero state matrix A_{aa} of the dynamical network in (1) and (2). These two characteristic polynomials are related as follows:*

$$P_{A_{aa}}(\lambda) = P_{A_{n_a}}(\lambda) + K_z P(\lambda), \quad (\text{A1})$$

where $P(\lambda)$ is a polynomial of order at most $n_a - (k+1)$ with leading coefficient equal to 1, k is the length of the decisive path, and K_z is a scalar multiplying the polynomial $P(\lambda)$.

Main Proof of Theorem 6: Consider that the edge with weight ϵ is located between vertex $m-1$ and m , and the distance from the vertex m and $m-1$ to the output is $d_{m,o}$ and $d_{m-1,o} = d_{m,o} + 1$ respectively.

We prove the result by showing that $K_z = \frac{1}{k+1} \text{trace}(P_{k+1} + N_{k+1})$ is $\mathcal{O}(\frac{1}{\epsilon})$ under the specified graph-theoretic conditions. To do so, we analyze the diagonal entries of the matrix $\Delta A_{n_a}^k$, $k = 0, 1, \dots$, which is present in the expressions for P_{k+1} and N_{k+1} . We note that the other matrices developed in Lemma 7 can be analyzed similarly.

The matrix $A_{n_a}^k$ captures information of the paths/walks of length k in the reduced graph G_1 . Thus, its entries will equal the weight product of these paths/walks, which cannot be $\mathcal{O}(\epsilon)$ since G_1 does not have any arc with

this weight, or zero. Hence, it is enough to study the entries of Δ . Let us first analyze an arbitrary entry of the perturbation matrix Δ . Expressing Δ in terms of the graph matrix (see [3]), we have that:

$$\Delta_{i,j} = \sum_{t=1}^{n_d} \sum_{r=2}^{n_d+1-t} \{A\}_{i,n_a+t} b_{n_d-t,r} \{A^{r-1}\}_{n,s}, \quad (\text{A2})$$

where $b_{i,j} = \frac{1}{\{A^{j-1}\}_{n,n-j+1}} \sum_{k=j+1}^{i+1} (\{A^k\}_{n,n-j+1} b_{i,k})$ for $i = 0, 1, \dots, d$ and $j = i, i-1, \dots, 2, 1$, is a recursion of $b_{i,i+1} = \frac{1}{\{A^i\}_{n,n-i}}$ and $\{A^i\}_{j,k}$ is the entry in the j^{th} row and the k^{th} column of the matrix A^i (see [3] for details).

We see that $b_{i,i+1}$ and $b_{i,j}$ are $\mathcal{O}(\frac{1}{\epsilon})$ if $i \geq d_{m-1,o} = d_{m,o} + 1$. Therefore, $\{A\}_{i,n_a+t} b_{n_d-t,r} = \mathcal{O}(\frac{1}{\epsilon})$ in (A2) if there is a vertex in the special input-output path, say vertex t , such that $d_{t,o} \geq d_{m,o} + 1$ and there is an arc from this vertex t to a vertex $i \in V_p$. We call this condition 1. This condition can be expressed as a distance from the input to the vertex t : $d_t \leq d - d_{m,o} - 1$. Condition 1 implies that $d_i \leq d - d_{m,o} = d_m$ for any vertex $i \in V_1$ that is connected to a vertex in the special input-output path.

Let suppose now that the vertex s is at a distance greater or equal to $d_{m-1,o} + 1$ from the output. We have then that $\{A^{r-1}\}_{n,s} = \mathcal{O}(\epsilon)$ for $r > d_{m,o} + 1$ and hence, in general, $\Delta_{i,s} \neq \mathcal{O}(\frac{1}{\epsilon})$. Therefore, $\Delta_{i,s} = \mathcal{O}(\frac{1}{\epsilon})$ if the vertex i is connected to a vertex in the special input-output path whose distance to the output is at most $d_{m,o} + 1$ and the vertex s is at distance less or equal to $d_{m,o} + 1$ and $d - d_i + 1$. Let us call this condition 2.

Let us now suppose that the vertex i is also the decisive vertex and satisfies condition 1 and 2. Clearly, $\{\Delta A_{n_a}^k\}_{i,i} = \mathcal{O}(\frac{1}{\epsilon})$, and hence (A2) becomes:

$$P_{A_{aa}}(\lambda) = P_{A_{n_a}}(\lambda) + \mathcal{O}\left(\frac{1}{\epsilon}\right) P(\lambda), \quad (\text{A3})$$

The equation (A3) can be viewed as root locus with gain parameter $\mathcal{O}(\frac{1}{\epsilon})$. Hence, if $k+1 \geq 3$ (condition 3), then for small enough ϵ at least two roots of $P_{A_{aa}}(\lambda)$ are in the CRHP. Therefore, for small enough ϵ , the finite zeros are in the CRHP if the distance from the decisive vertex i to its pair j is $k \geq 2$. ■

At its essence, the proof of this theorem depends on tracing the finite zeros of the system in terms of the weak edge weight ϵ . This trace can be approximated by a root locus, which becomes increasingly accurate in the limit of small ϵ . In Fig. 3, we present the trace of the zeros versus ϵ and compare to the root locus to give insight into the proof of Theorem 6.

Fig. 3. For the example network dynamics (Fig. 1 in Section 4.3), the finite zeros are traced as a function of the weak edge weight ϵ (Left). The root-locus approximation of the finite zeros is also given (Right), which is needed to prove Theorem 6.

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